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THE ESTIMATION OF MAGNETOTELLURIC IMPEDANCE TENSOR ELEMENTS FROM MEASURED DATA†

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Six different estimates of the magnetotelluric impedance tensor elements may be computed from measured data by use of auto-power and cross-power density spectra. We show that each of the estimates satisfies a mean-square error criterion. Two of the six estimates are relatively unstable in the one-dimensional case when the incident fields are unpolarized. For the remaining four estimates, it is shown that two are unaffected by random noise on the H signal, but are biased upward by random noise on the E signal. The remaining two estimates are unaffected by random noise on the E signal, but are biased downward by random noise on the H signal. Computation of all of the estimates provides a measure of the total amount of noise present, as indicated by a stability coefficient for the estimates. In the absence of additional information as to the relative signal-to-noise ratios of the E and H signals, we suggest that a mean estimate be used. A numerical example is included.

INTRODUCTION

The magnetotelluric sounding method for the determination of subsurface electrical conductivity profiles as proposed by Cagniard (1953) is based upon the assumption of a horizontally stratified layered earth model. For an anisotropic or laterally inhomogeneous earth, the impedance becomes a tensor quantity. Methods for the estimation of the impedance tensor elements have been studied and proposed by many investigators, including Neves (1957), Cantwell (1960), Bostick and Smith (1962), Swift (1967), and Rankin and Reddy (1969). Their methods are generally quite similar and use power spectral density estimates of orthogonal E and H field data.

In the formulation employed both by Swift (1967) and by Rankin and Reddy (1969, equations 16a–16d), the authors recognized that the least square estimation being used would be insensitive to independent noises on the E field, but would bias the principal impedances to the low side for noises on the H-field data. It is the purpose of this paper to point out other least square estimators which will have the opposite effect, i.e., which will be insensitive to indepen dent noises on the H field and produce principal impedance values that are biased to the high side for noise on the E field. We suggest that the computation of several estimates will indicate when noise problems are severe and will serve as a guide for selection of estimates, which, under certain conditions, will have relatively less bias.

THE ESTIMATION PROBLEM

Consider the equation

$$E_x = Z_{xx}H_x + Z_{xy}H_y,$$

where E_x, H_x, and H_y may be considered to be Fourier transforms of measured electric and magnetic field data. If one has two independent measurements of E_x, H_x, and H_y at a given frequency, denoted by $E_{x1}, H_{x1}, H_{y1}, E_{x2}, H_{x2}$, and H_{y2} respectively,

$$Z_{xx} = \frac{\begin{vmatrix} E_{x1} & H_{y1} \\ E_{x2} & H_{y2} \end{vmatrix}}{\begin{vmatrix} H_{x1} & H_{y1} \\ H_{x2} & H_{y2} \end{vmatrix}} \qquad \text{(1a)}$$

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and

$$Z_{xy} = \frac{\begin{vmatrix} H_{x1} & E_{x1} \\ H_{x2} & E_{x2} \end{vmatrix}}{\begin{vmatrix} H_{x1} & H_{y1} \\ H_{x2} & H_{y2} \end{vmatrix}}, \tag{1b}$$

provided

$$H_{x1}H_{y2} - H_{x2}H_{y1} \neq 0.$$

Equation (1) simply states that the two field measurements must have different source polarizations. If the two have the same polarization, they are not independent.

Since any physical measurement of E or H will include some noise, it is usually desirable to make more than two independent measurements, and then to use some type of averaging that will reduce the effects of the noise. Suppose one has n measurements of E_x, H_x, and H_y at a given frequency. One can then estimate Z_{xx} and Z_{xy} in the mean-square sense; that is, one may define

$$\psi = \sum_{i=1}^{n} (E_{xi} - Z_{xx}H_{xi} - Z_{xy}H_{yi}) \cdot (E_{xi}^* - Z_{xx}^*H_{xi}^* - Z_{xy}^*H_{yi}^*), \tag{2}$$

where E_{xi}^* is the complex conjugate of E_{xi}, etc., and then find the values of Z_{xx} and Z_{xy} that minimize ψ. Setting the derivatives of ψ with respect to the real and imaginary parts of Z_{xx} to zero yields

$$\sum_{i=1}^{n} E_{xi}H_{xi}^* = Z_{xx} \sum_{i=1}^{n} H_{xi}H_{xi}^* + Z_{xy} \sum_{i=1}^{n} H_{yi}H_{xi}^*. \tag{3}$$

Similarly, setting the derivatives of ψ with respect to the real and imaginary parts of Z_{xy} to zero yields

$$\sum_{i=1}^{n} E_{xi}H_{yi}^* = Z_{xx} \sum_{i=1}^{n} H_{xi}H_{yi}^* + Z_{xy} \sum_{i=1}^{n} H_{yi}H_{yi}^*. \tag{4}$$

Notice that the summations represent auto-power and cross-power density spectra. Equations (3) and (4) may be solved simultaneously for Z_{xx} and Z_{xy}. This solution will minimize the error caused by noise on E_x. It is possible to define other mean-square estimates that minimize other types of noise. For example, if one takes

$$\psi = \sum_{i=1}^{n} \left(\frac{E_{xi}}{Z_{xx}} - H_{xi} - \frac{Z_{xy}}{Z_{xx}} H_{yi}\right) \cdot \left(\frac{E_{xi}^*}{Z_{xx}^*} - H_{xi}^* - \frac{Z_{xy}^*}{Z_{xx}^*} H_{yi}^*\right), \tag{5}$$

the resulting solution will minimize the error introduced by noise on H_x.

There are four distinct equations that arise from the various mean-square estimates. In terms of the auto-power and cross-power density spectra, they are

$$\overline{E_xE_x^*} = Z_{xx}\overline{H_xE_x^*} + Z_{xy}\overline{H_yE_x^*}, \tag{6}$$

$$\overline{E_xE_y^*} = Z_{xx}\overline{H_xE_y^*} + Z_{xy}\overline{H_yE_y^*}, \tag{7}$$

$$\overline{E_xH_x^*} = Z_{xx}\overline{H_xH_x^*} + Z_{xy}\overline{H_yH_x^*}, \tag{8}$$

and

$$\overline{E_xH_y^*} = Z_{xx}\overline{H_xH_y^*} + Z_{xy}\overline{H_yH_y^*}. \tag{9}$$

Strictly speaking, equations (6) through (9) are valid only if $\overline{E_xE_x^*}$, $\overline{E_xE_y^*}$, etc., represent the power density spectra at a discrete frequency ω. In practice, however, the Z_{ij} are slowly varying functions of frequency; consequently, $\overline{E_xE_x^*}$, etc., may be taken as averages over some finite bandwidth. This is fortunate, since it facilitates estimation of the power density spectra.

ESTIMATION OF Z FROM POWER DENSITY SPECTRA

Consider again equations (6) through (9). Under certain conditions, these equations are independent, so that any two of them may be solved simultaneously for Z_{xx} and Z_{xy}. Since there are six possible distinct pairs of equations, there are six ways to estimate Z_{xx} and Z_{xy}. For example, the six estimates for Z_{xy} are

$$Z_{xy} = \frac{(\overline{H_xE_x^*})(\overline{E_xE_y^*}) - (\overline{H_xE_y^*})(\overline{E_xE_x^*})}{(\overline{H_xE_x^*})(\overline{H_yE_y^*}) - (\overline{H_xE_y^*})(\overline{H_yE_x^*})}, \tag{10}$$

$$\overline{Z}_{xy}=\frac{\overline{(H_xE_x^*)}\,\overline{(E_xH_x^*)}-\overline{(H_xH_x^*)}\,\overline{(E_xE_x^*)}}{\overline{(H_xE_x^*)}\,\overline{(H_yH_x^*)}-\overline{(H_xH_x^*)}\,\overline{(H_yE_x^*)}}, \quad (11)$$

$$\overline{Z}_{xy}=\frac{\overline{(H_xE_x^*)}\,\overline{(E_xH_y^*)}-\overline{(H_xH_y^*)}\,\overline{(E_xE_x^*)}}{\overline{(H_xE_x^*)}\,\overline{(H_yH_y^*)}-\overline{(H_xH_y^*)}\,\overline{(H_yE_x^*)}}, \quad (12)$$

$$\overline{Z}_{xy}=\frac{\overline{(H_xE_y^*)}\,\overline{(E_xH_x^*)}-\overline{(H_xH_x^*)}\,\overline{(E_xE_y^*)}}{\overline{(H_xE_y^*)}\,\overline{(H_yH_x^*)}-\overline{(H_xH_x^*)}\,\overline{(H_yE_y^*)}}, \quad (13)$$

$$\overline{Z}_{xy}=\frac{\overline{(H_xE_y^*)}\,\overline{(E_xH_y^*)}-\overline{(H_xH_y^*)}\,\overline{(E_xE_y^*)}}{\overline{(H_xE_y^*)}\,\overline{(H_yH_y^*)}-\overline{(H_xH_y^*)}\,\overline{(H_yE_y^*)}}, \quad (14)$$

and

$$\overline{Z}_{xy}=\frac{\overline{(H_xH_x^*)}\,\overline{(E_xH_y^*)}-\overline{(H_xH_y^*)}\,\overline{(E_xH_x^*)}}{\overline{(H_xH_x^*)}\,\overline{(H_yH_y^*)}-\overline{(H_xH_y^*)}\,\overline{(H_yH_x^*)}}, \quad (15)$$

where $\overline{Z}_{xy}$ denotes a measured estimate of Z_{xy}. It turns out that two of these expressions tend to be relatively unstable for the one-dimensional case, particularly when the incident fields are unpolarized. For that case, $\overline{E_xE_y^*}$, $\overline{E_xH_x^*}$, $\overline{E_yH_y^*}$ and $\overline{H_xH_y^*}$ tend toward zero, so that equations (12) and (13) become indeterminant. The other four expressions are quite stable and correctly predict $Z_{xy}=E_x/H_y$ for the one-dimensional case, provided the incident fields are not highly polarized.

These same remarks are true for the other three impedance elements Z_{xx}, Z_{yx}, and Z_{yy}. In each case, there are six ways to estimate Z_{ij}, two of which are unstable for one-dimensional models with unpolarized incident fields. Also, in each case, the other four estimates are quite stable for any reasonable earth model, provided the incident fields are not highly polarized.

As was mentioned earlier, any physical measurement of E or H will necessarily contain some noise. It is desirable now to consider how such noise will affect the Z estimates defined above. Suppose that

$$E_x = E_{xs} + E_{xn}, \quad (16)$$

$$E_y = E_{ys} + E_{yn}, \quad (17)$$

$$H_x = H_{xs} + H_{xn}, \quad (18)$$

and

$$H_y = H_{ys} + H_{yn}, \quad (19)$$

where

$$\begin{bmatrix} E_{xs} \\ E_{ys} \end{bmatrix} = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} \begin{bmatrix} H_{xs} \\ H_{ys} \end{bmatrix}$$

and E_{xn}, E_{yn}, H_{xn}, and H_{yn} are noise terms. If the noise terms are all zero, the four stable estimates of each of the elements of Z are the same; and

$$\overline{Z}_{ij} = Z_{ij}.$$

On the other hand, when the noise terms are nonzero, the four estimates are, in general, different.

Equation (15) for Z_{xy} corresponds to the one that Swift (1967) used. He showed that his estimates of Z_{ij} were biased down by random noise on the H signal but were not affected by random noise on the E signal. Similar arguments for the four stable estimates defined above indicate that, in each case, two of the estimates are biased down by random noise on H and are not biased by random noise on E [for example, equations (14) and (15) for $\overline{Z}_{xy}$], while the other two are biased up by random noise on E and are not biased by random noise on H [for example, equations (10) and (11) for $\overline{Z}_{xy}$]. The effects of the noise are most easily seen for the one-dimensional model. For this model, if the incident fields are depolarized so that $\overline{E_xE_y^*}$, $\overline{E_xH_x^*}$, and $\overline{H_xH_y^*}$ tend to zero, equations (10) and (11) for $\overline{Z}_{xy}$ reduce to

$$\overline{Z}_{xy} = \overline{E_xE_x^*}/\overline{H_yE_x^*}. \quad (20)$$

Equations (14) and (15) reduce to

$$\overline{Z}_{xy} = \overline{E_xH_y^*}/\overline{H_yH_y^*}. \quad (21)$$

If one assumes that E_x and H_y are given by equations (16) and (19) and the E_{yn} and H_{xn} are random and independent of the signals and of each other, the expected values of the power density spectra are

$$\langle\overline{E_xE_x^*}\rangle = \langle\overline{E_{xs}E_{xs}^*}\rangle + \langle\overline{E_{xn}E_{xn}^*}\rangle,$$

$$\langle\overline{H_yH_y^*}\rangle = \langle\overline{H_{ys}H_{ys}^*}\rangle + \langle\overline{H_{yn}H_{yn}^*}\rangle,$$

and

$$\langle\overline{E_xH_y^*}\rangle = \langle\overline{H_yE_x^*}\rangle = \langle\overline{E_{xs}H_{ys}^*}\rangle.$$

Thus, if the spectral estimates contain enough terms in the average so that the cross terms may

Table 1. Example of estimates of $|\bar{Z}_{xy}|$ from magnetotelluric data near Pecos, Texas

Estimate no.	Center freq. hz	$\lvert\bar{Z}_{xy}\rvert$ from Eq. (10)	$\lvert\bar{Z}_{xy}\rvert$ from Eq. (11)	$\lvert\bar{Z}_{xy}\rvert$ from Eq. (14)	$\lvert\bar{Z}_{xy}\rvert$ from Eq. (15)	Mean $\lvert\bar{Z}_{xy}\rvert$	Mean $\bar{\rho}_{xy}$	Stability coefficient
1	0.00033	0.146	0.173	0.118	0.116	0.137	11.4	0.539
2	0.00066	0.234	0.241	0.202	0.203	0.219	14.7	0.725
3	0.00098	0.316	0.322	0.287	0.288	0.303	18.6	0.813
4	0.00131	0.360	0.366	0.336	0.336	0.349	18.6	0.859
5	0.00164	0.415	0.419	0.409	0.409	0.413	20.8	0.965
6	0.00197	0.430	0.433	0.427	0.428	0.429	18.7	0.984
7	0.00230	0.442	0.443	0.435	0.435	0.439	16.8	0.968
8	0.00263	0.454	0.453	0.445	0.445	0.449	15.4	0.964
9	0.00295	0.485	0.491	0.468	0.470	0.478	15.5	0.927
10	0.00328	0.497	0.505	0.479	0.482	0.491	14.7	0.920
11	0.00361	0.495	0.501	0.486	0.487	0.492	13.4	0.955
12	0.00394	0.493	0.499	0.485	0.486	0.491	12.2	0.958
13	0.00427	0.486	0.490	0.479	0.480	0.484	11.0	0.965
14	0.00459	0.512	0.516	0.501	0.502	0.508	11.2	0.954
15	0.00492	0.528	0.533	0.512	0.513	0.521	11.1	0.934
16	0.00525	0.529	0.536	0.512	0.514	0.523	10.4	0.930
17	0.00558	0.553	0.561	0.539	0.541	0.548	10.8	0.941
18	0.00591	0.585	0.597	0.571	0.575	0.581	11.5	0.940
19	0.00623	0.594	0.608	0.578	0.582	0.590	11.2	0.931
20	0.00656	0.585	0.600	0.561	0.565	0.577	10.2	0.903

be neglected (i.e., $\overline{E_{xs}E_{xn}^*}$, etc., are negligible), equation (20) gives

$$\bar{Z}_{xy} = \frac{\overline{E_{xs}E_{xs}^*} + \overline{E_{xn}E_{xn}^*}}{\overline{H_{ys}E_{xs}^*}} = Z_{xy}\left(1 + \frac{E \text{ noise power}}{E \text{ signal power}}\right) \tag{22}$$

and equation (21) gives

$$\bar{Z}_{xy} = \frac{\overline{E_{xs}H_{ys}^*}}{\overline{H_{ys}H_{ys}^*} + \overline{H_{yn}H_{yn}^*}} = Z_{xy} \Big/ \left(1 + \frac{H \text{ noise power}}{H \text{ signal power}}\right). \tag{23}$$

Hence, the estimate shown in equation (22) is biased to the high side by random noise on E, while the one in equation (23) is biased to the low side by random noise on H. For similar percentages of random noise on E and H, an average of the various estimates hopefully will be better than any one estimate by itself. Also, the scatter between the various estimates should be a good measure of the amount of random noise present.

In practice, of course, things are not quite this neat because the assumption that the cross terms in the average power estimates are negligible may not be valid. For example, terms of the form $\overline{E_{xn}H_{yn}^*}$ will not be negligible if the two noises are coherent. Such might be the case for certain types of instrumentation noise or local industrial noise or 60 hz power line noise. Also, terms of the form $\overline{E_{xs}E_{xn}^*}$ will not be negligible if the noise is coherent with the signal source. Even if all of the noise terms are random and independent of the signals and of each other, the cross terms may not be negligible if the average power estimates do not have enough degrees of freedom.

NUMERICAL EXAMPLE

A numerical example of the computation of the estimate of the magnitude of one impedance element $|\bar{Z}_{xy}|$ is shown in Table 1. Only a portion of one low frequency recording taken near Pecos, Texas is included in this example. This recording was approximately 6 hours long and was sampled every 30.47 sec. The center frequency for each of the first 20 bands is given in the table, along with the estimates of $|\bar{Z}_{xy}|$ computed from equations (10), (11), (14), and (15). Also computed were the mean value of $|\bar{Z}_{xy}|$, the corresponding mean resistivity estimate from

$$\bar{\rho}_{xy} = \frac{1}{\omega\mu} \mid \bar{Z}_{xy} \mid^2,$$

and a stability coefficient s from

$$s = \frac{[\,|\bar{Z}_{xy}|\ (\text{from eq. }(11))][\,|\bar{Z}_{xy}|\ (\text{from eq. }(12))]}{[\,|\bar{Z}_{xy}|\ (\text{from eq. }(14))][\,|\bar{Z}_{xy}|\ (\text{from eq. }(15))]}.$$

The stability coefficient will be unity if the four estimates are identical and will decrease as the estimates diverge.

Estimate no. 6 in Table 1 has the highest value of the stability coefficient, $s=0.984$, and a correspondingly low variation in the four estimates for $|\bar{Z}_{xy}|$. The low value, $s=0.539$, for the lowest frequency band is a typical result for power spectrum estimates based on a small number of degrees of freedom. The number of degrees of freedom is approximately 14 for this frequency band; a longer sample of data would be expected to yield more reliable estimates.

It should be noted, as expected, that the set of estimates computed from equations (10) and (11) (biased upward by E noise) are higher in every case than the set computed from equations (14) and (15) (biased downward by H noise). Lacking further evidence, however, we can not say that either set is unbiased by noise. If they are not, the mean estimate is likely the most reliable. The mean estimate does have a tendency to produce a smoother curve for $\bar{\rho}_{xy}$ than do values taken from either set.

If additional noise measurements on the equipment establish that in a certain frequency band the E or H signal-to-noise ratios are significantly different, computation of $\bar{\rho}_{xy}$ should be made with the set with the higher signal-to-noise ratio. Knowledge of the magnitude of the individual E and H power spectra in each frequency band will help in this regard, but noise information is also needed.

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