

On the finite difference solution of two-dimensional induction problems

C. R. Brewitt-Taylor and J. T. Weaver *Department of Physics,
University of Victoria, Victoria, British Columbia, Canada*

Received 1976 March 29; in original form 1976 February 16

Summary. The numerical solution by finite differences of two-dimensional problems in electromagnetic induction is re-examined with a view to generalizing the method to three-dimensional models. Previously published work, in which fictitious values were used to derive the finite difference equations, is discussed and some errors in the theory which appear to have gone undetected so far, are pointed out. It is shown that the previously published *B*-polarization formulas are incorrect at points where regions of different conductivity meet, and that the *E*-polarization formulas are inaccurate when the step sizes of the numerical grid around the point are uneven. An appropriately-modified version of the two-dimensional theory is developed on the assumption that the Earth's conductivity is a smoothly-varying function of position, a method which naturally lends itself to three-dimensional generalization. All the required finite-difference formulas are derived in detail, and presented in a form which is suitable for programming. A simple numerical calculation is given to illustrate the application of the method and the results are compared with those obtained from previous work.

1 Introduction

The solution of two-dimensional induction problems by numerical methods has received widespread attention during the last several years. The various techniques that have been used (transmission line analogy, integral equation, finite-element and finite-difference methods) have recently been reviewed by Jones (1973) and by Ward, Peeples & Ryu (1973). In many of the published papers, authors have emphasized the presentation of numerical results for particular models rather than the development of the algebraic equations they have solved, with the result that it has not always been possible to acquire a detailed knowledge of their procedures. The outstanding exceptions are Jones & Pascoe (1971) and Pascoe & Jones (1972) who have not only published the theory but also the FORTRAN program of their finite-difference method for solving the problem of induction in a general two-dimensional structure embedded in a layered earth. By doing this they have generously enabled other researchers to use the program directly for solving particular induction problems, but as a result they have exposed the algebraic details of their own work to greater

scrutiny than that of other authors. An error in their finite difference representation of second derivatives for variable grid spacings was discovered by Williamson, Hewlett & Tamemagi (1974). Jones & Thomson (1974) have shown that its effect on calculated results is much reduced when a numerical grid whose spacings are not too irregular is used.

Research activity in this field is now directed towards the solution of three-dimensional problems. An elegant and economical formulation of this problem in terms of a vector integral equation has been published by Raiche (1974), and independently by Weidelt (1975) who has described and applied the method in some detail. Lines & Jones (1973a,b) have derived a three-dimensional generalization of the finite difference method of solving for the electric vector and have applied it to models representing islands located near a long coastline. Such a model approaches a straight-forward coastline problem at large distances from the island, so that two-dimensional problems still have to be solved in order to first determine the boundary conditions for the three-dimensional model.

It is for this reason that in formulating a finite difference method for solving three-dimensional models in terms of the magnetic rather than the electric vector we have had occasion to re-examine the two-dimensional theory. In doing so we believe we have uncovered some misconceptions and errors (other than the one pointed out by Williamson *et al.* (1974)) in previously published work.

In this paper we shall discuss these points of difficulty in detail and show one way of overcoming them. This leads to a new finite-difference formulation of the general two-dimensional induction problem which can be used as an alternative to finite element and other methods in which the difficulties mentioned do not arise.

2 The equations of the electromagnetic field

The basic equations of electromagnetic induction in a region of conductivity σ and vacuum permeability μ_0 are (in SI units)

$$\text{curl } \mathbf{E} = -i\omega\mathbf{B} \quad (2.1)$$

$$\text{curl } \mathbf{B} = \mu_0\sigma\mathbf{E} \quad (2.2)$$

where $\mathbf{E} \exp(i\omega t)$ and $\mathbf{B} \exp(i\omega t)$ are respectively the electric and magnetic field vectors varying in time t with angular frequency ω . It follows from (2.1) and (2.2) that \mathbf{E} and \mathbf{B} satisfy the respective differential equations

$$\text{curl curl } \mathbf{E} = -i\kappa\mathbf{E} \quad (2.3)$$

$$\rho\nabla^2\mathbf{B} - \text{grad } \rho \times \text{curl } \mathbf{B} = i\mathbf{B} \quad (2.4)$$

where $\kappa = 1/\rho = \omega\mu_0\sigma$.

Let the unit vectors \mathbf{x} , \mathbf{y} , \mathbf{z} define a rectangular Cartesian coordinate system in which, for a two-dimensional problem, it may be assumed that σ , \mathbf{E} and \mathbf{B} are independent of the variable x . It is well known that a general two-dimensional field of this type separates into (i) an E -polarization field,

$$\mathbf{E} = E\mathbf{x}, \quad \mathbf{B} = (i/\omega)(\mathbf{y} \partial E/\partial z - \mathbf{z} \partial E/\partial y) \quad (2.5)$$

and (ii) a B -polarization field

$$\mathbf{B} = B\mathbf{x}, \quad \mathbf{E} = \omega\rho(\mathbf{y} \partial B/\partial z - \mathbf{z} \partial B/\partial y) \quad (2.6)$$

and that these two cases may be considered independently. For E -polarization, equation (2.3) reduces to

$$\partial^2 E / \partial y^2 + \partial^2 E / \partial z^2 = i\kappa E \quad (2.7)$$

and for B -polarization equation (2.4) can be written as

$$\rho \frac{\partial^2 B}{\partial y^2} + \rho \frac{\partial^2 B}{\partial z^2} + \frac{\partial \rho}{\partial y} \frac{\partial B}{\partial y} + \frac{\partial \rho}{\partial z} \frac{\partial B}{\partial z} = iB. \quad (2.8)$$

In a non-conducting region ($\sigma = 0$), equation (2.2) may be replaced by the simpler equation

$$\mathbf{B} = -\text{grad } \Omega \quad (2.9)$$

where, by equation (2.1)

$$\nabla^2 \Omega = 0. \quad (2.10)$$

In E -polarization, equations (2.5) and (2.9) imply

$$\partial E / \partial y = -i\omega \partial \Omega / \partial z, \quad \partial E / \partial z = i\omega \partial \Omega / \partial y \quad (2.11)$$

while in B -polarization equations (2.6) and (2.9) yield

$$B = H(\text{const}). \quad (2.12)$$

We shall be concerned with the problem of solving equations (2.7) and (2.8) in the half-space $z > 0$ for a given function κ corresponding to the conductivity distribution within the Earth. The region $z < 0$ above the Earth's surface will be taken as non-conducting, and we shall assume that the inducing magnetic field is horizontal and uniform over the entire region.

3 Boundary conditions

We make the basic assumption that σ becomes a function of z alone as $|y| \rightarrow \infty$. Thus in both polarizations the problem reduces at large horizontal distances to one of induction by a uniform, horizontal magnetic field in an earth whose conductivity varies only with depth. It is a well-known property of this one-dimensional problem (e.g. Jones & Price 1970) that the *total* magnetic field above the conductor is horizontal, uniform and independent of the conductivity distribution in the Earth. Since the inducing field is the same everywhere it follows that the total magnetic field must have the same constant limiting value H as $y \rightarrow \pm \infty$ in $z < 0$. Also, in the region $z > 0$, $\mathbf{B} \rightarrow 0$ as $z \rightarrow \infty$. The development of the remaining boundary conditions is different for the two polarizations and we shall treat them separately.

E -POLARIZATION

Let $E \rightarrow E^\pm$ and $\kappa \rightarrow \kappa^\pm$ as $y \rightarrow \pm \infty$ in the region $z > 0$. Then according to equation (2.7), E^\pm are the solutions of

$$\partial^2 E^\pm / \partial z^2 = i\kappa^\pm E^\pm \quad (3.1)$$

subject to the boundary conditions $E^\pm \rightarrow 0$ as $z \rightarrow \infty$, and

$$(\partial E^\pm / \partial z)_{z=0} = -i\omega H, \quad (3.2)$$

this last condition following from equation (2.5) and the continuity of the magnetic field at $z = 0$. Equation (3.1) can be solved by standard methods either analytically or numerically. Only if $\kappa^+ = \kappa^-$ will $E^+ = E^-$.

The influence of conductivity variations will also vanish at large distances in the region $z < 0$. The general solution of equation (2.1) in $z < 0$ subject to the condition $\Omega \rightarrow -Hy$ as $z \rightarrow -\infty$ is

$$\Omega + Hy = \int_{-\infty}^{\infty} \frac{f(\nu)}{\nu} \exp(z|\nu| - i\nu y) d\nu \quad (3.3)$$

where $\sqrt{2\pi}f(\nu)/\nu$ is the Fourier transform of $\Omega + Hy$ at $z = 0$. Defined in this way, with the factor $1/\nu$ removed, f is a suitably well-behaved function such that the Riemann–Lebesgue lemma can be used to ensure that the boundary condition $\partial\Omega/\partial y \rightarrow -H$ as $|y| \rightarrow \infty$ is automatically satisfied. It follows from equation (2.11) and some algebraic manipulation that

$$\partial E/\partial y = i\omega \int_0^{\infty} \{f(-\nu) \exp(i\nu y) - f(\nu) \exp(-i\nu y)\} \exp(z\nu) d\nu \quad (3.4)$$

$$\partial E/\partial z = -i\omega H + \omega \int_0^{\infty} \{f(-\nu) \exp(i\nu y) + f(\nu) \exp(-i\nu y)\} \exp(z\nu) d\nu.$$

Assuming that f can be expanded in a Maclaurin series on either side of $\nu = 0$, we can integrate term by term to obtain an asymptotic expansion of the integrals (3.4) for large $r = (y^2 + z^2)^{1/2}$. Neglecting terms $O(1/r^2)$ we find that

$$\begin{aligned} \partial E/\partial y &\sim iC_- z/r^2 - C_+ y/r^2 \\ \partial E/\partial z &\sim -i\omega H - C_+ z/r^2 - iC_- y/r^2 \end{aligned} \quad (3.5)$$

where $C_\pm = \omega\{f(+0) \pm f(-0)\}$. Since $\partial E/\partial y \rightarrow 0$ as $|y| \rightarrow \infty$, it follows at once that $C_+ = 0$ and hence that

$$E \sim A - i\omega Hz - iC_- \arctan(y/|z|) \quad (3.6)$$

where A is a constant of integration. Now as $y \rightarrow \pm\infty$ on $z = 0$, $\arctan(y/|z|) \rightarrow \pm\frac{1}{2}\pi$ and $E \rightarrow (E^\pm)_{z=0}$, these last values being given by the solution of equation (3.1). Substituting these conditions in equation (3.6), eliminating A and C_- , and defining

$$\bar{E} = \frac{1}{2}(E^+ + E^-)_{z=0}, \quad \Delta E = (E^+ - E^-)_{z=0} \quad (3.7)$$

we deduce that

$$E \sim \bar{E} - i\omega Hz + (\Delta E/\pi) \arctan(y/|z|). \quad (3.8)$$

This serves as an adequate approximation to the field at large distances $(y^2 + z^2)^{1/2}$ in the region $z < 0$, and completes the boundary conditions required for solving E -polarization problems.

It should be noted that equation (3.8) differs slightly from the corresponding one derived

by Jones & Price (1970) using rather different arguments. They considered a rectangular boundary enclosing the region $z < 0$ and assumed that the vertical magnetic field (i.e. $\partial E/\partial y$) is constant along the upper boundary $z = -R$ for large R . In our equation, however, $\partial E/\partial y$ has a small variation with y on $z = -R$, which is probably an indication of the increasing accuracy of our asymptotic formula towards the upper corners of a rectangular boundary. Equation (3.8) should really be applied around the semi-circular boundary $r = R$. Nevertheless near $z = 0$ and $y = 0$ it should agree exactly with the Jones & Price formula for the rectangular boundary $y = \pm R$, ($0 < z < -R$) and $z = -R$, ($|y| < R$). As $z \rightarrow -0$, $y \rightarrow \pm R$, equation (3.8) reduces to

$$E \sim (E^\pm)_{z=0} - i\omega Hz \tag{3.9}$$

and as $|y| \rightarrow 0$, $z \rightarrow -R$, it becomes

$$E \sim \bar{E} + i\omega HR + y \Delta E/\pi R. \tag{3.10}$$

Equation (3.9) is exactly the same as the corresponding Jones & Price boundary conditions (their equations (20) and (21)), but equation (3.10) differs from theirs (equation (27) with $k = h_0 = R$) in the third term where they have a factor $\frac{1}{2}$ in place of our $1/\pi$. (Actually they also omitted the second term in equation (3.10) but this appears to be a minor slip resulting from a wrong substitution for E at $y = \pm k$.)

Another way of dealing with equations (3.4) is to apply a negative Hilbert transform

$$K[g(y)] \equiv -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(u)}{u-y} du. \tag{3.11}$$

to the function $\partial E/\partial y$. It is simple to show by repeated integration that

$$K[\partial E/\partial y] = i\omega H + \partial E/\partial z \tag{3.12}$$

so that, in particular, at $z = 0$ we have the boundary condition

$$(\partial E/\partial z)_{z=0} = -i\omega H + K[(\partial E/\partial y)_{z=0}]. \tag{3.13}$$

This serves as an integral boundary condition equivalent to the one devised by Schmucker (1971), with K representing the operator introduced by Kertz (1954). It has the advantage of effectively removing the region $z < 0$ from further consideration but complicates the finite difference formulation of the problem somewhat by relating the field at a point on the surface not just to the field at four neighbouring grid points but to the field at all the other grid points on $z = 0$. We shall not use the integral boundary condition in this paper but it is interesting to see that both it and the Jones-Price type of boundary condition are different manifestations of the same integrals (3.4).

B-POLARIZATION

In this polarization we need only consider the region $z > 0$ with $(B)_{z=0} = H(\text{const})$ as the boundary condition along the surface according to equation (2.12). For side boundary conditions we have $B \rightarrow B^\pm$ as $y \rightarrow \pm\infty$ where B^\pm are solutions of

$$\rho^\pm \frac{\partial^2 B^\pm}{\partial z^2} + \frac{\partial \rho^\pm}{\partial z} \frac{\partial B^\pm}{\partial z} = iB^\pm \tag{3.14}$$

subject to $(B^\pm)_{z=0} = H$.

4 Discussion of finite difference methods

Equations (2.7) and (2.8) can be solved numerically by finite difference methods. In this section we shall consider a mesh whose grid points have an equal separation h in both the y - and z -directions. Variable grid separations could be easily incorporated in the theory but at this point they would only distract from the main theme of the discussion.

Consider a typical grid point 0 and its neighbouring points 1–4 in a region of uniform conductivity, as shown in Fig. 1(a). Since $\text{grad } \rho = 0$, both equations (2.7) and (2.8) have the same finite difference representation

$$F_1 + F_2 + F_3 + F_4 = (4 + ikh^2) F_0 \quad (4.1)$$

with an error $O(h^3)$, where F stands for either E or B and the subscript indicates the grid-point at which the field is evaluated.

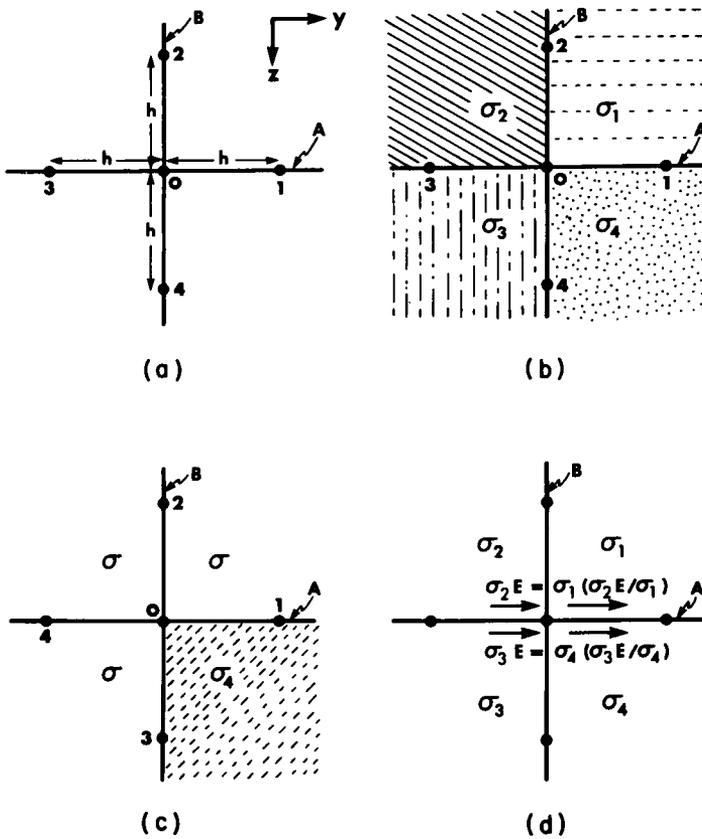


Figure 1. A typical node.

Jones & Pascoe (1971) considered instead the most general configuration possible, as shown in Fig. 1(b) where the point 0 is at the junction of four regions of different conductivities, σ_1 , σ_2 , σ_3 , and σ_4 . They envisaged these different regions as being sharply separated by the grid lines A and B through point 0, and sought the appropriate finite difference equation by introducing 'fictitious' values and by then applying the usual electromagnetic boundary conditions across the sharp boundaries.

E-POLARIZATION

Let us examine this procedure for E -polarization. If $E_i^{(j)}$ represents the fictitious field value at the point i which is associated with region j , then we may write down the four finite difference equations (4.1) which apply at the point 0 as follows

$$\begin{aligned} E_1 + E_2 + E_3^{(1)} + E_4^{(1)} &= (4 + i\kappa_1 h^2) E_0 \\ E_1^{(2)} + E_2 + E_3 + E_4^{(2)} &= (4 + i\kappa_2 h^2) E_0 \\ E_1^{(3)} + E_2^{(3)} + E_3 + E_4 &= (4 + i\kappa_3 h^2) E_0 \\ E_1 + E_2^{(4)} + E_3^{(4)} + E_4 &= (4 + i\kappa_4 h^2) E_0. \end{aligned} \quad (4.2)$$

The uniqueness of the field values at each grid point is ensured by the continuity of the (tangential) electric field across the boundaries. The continuity of the tangential magnetic field components provides additional equations. For example, from equation (2.5) we see that $\partial E/\partial z$ is continuous across line A joining points 3 and 1. Using the central difference formula for the derivatives at the point 0, we find that

$$E_4^{(1)} - E_2 = E_4 - E_2^{(4)}, \quad E_4^{(2)} - E_2 = E_4 - E_2^{(3)} \quad (4.3)$$

again with an error $O(h^3)$. Similarly, the relations

$$E_1^{(2)} - E_3 = E_1 - E_3^{(1)}, \quad E_1^{(3)} - E_3 = E_1 - E_3^{(4)} \quad (4.4)$$

follow from the continuity of the tangential magnetic field across the line B joining points 2 and 4.

We now have eight equations for the eight unknown fictitious values. However, it is easily verified that the determinant of the coefficients vanishes, the rank of the coefficient matrix being 7. Thus the equations have a solution if and only if the rank of the augmented matrix is also 7, and this means that the values E_i ($i = 0, 1, 2, 3, 4$) must be connected to each other in a certain way. The required condition is most easily obtained by direct algebraic elimination of the fictitious values. We find that

$$E_1 + E_2 + E_3 + E_4 = (4 + ih^2\kappa_0) E_0 \quad (4.5)$$

where $\kappa_0 = \frac{1}{4}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)$ is the average value of κ in the four surrounding regions.

Jones & Pascoe (1971) also obtained equation (4.5) but by a different procedure which involved an application of the boundary conditions using one-sided rather than central difference formulas as approximations to the first derivatives. Apart from the fact that the one-sided differences have errors $O(h^2)$ which are as great as terms retained elsewhere in the analysis, their use also meant that the boundary conditions on the tangential magnetic field were, in effect, doubly applied at each boundary. Thus instead of the four equations (4.3) and (4.4) Jones & Pascoe obtained eight separate equations which, incidentally, contained the tacit implication $E_i^{(j)} = E_i$ ($i = 1, 2, 3, 4$). Moreover, when taken together with equations (4.2) these eight new equations gave rise to an over-determined system which the authors made determinate by adding equations (4.2) together to yield a single equation in place of the previous four. Although this is not a valid mathematical procedure it is, in fact, equivalent to averaging the four conductivity values and so leads directly to equation (4.5) which does have a valid physical interpretation.

Unfortunately our own derivation of equation (4.5) is also open to criticism for we have so far overlooked the remaining boundary condition which states that the normal compon-

ent of \mathbf{B} must be continuous. It is easily verified that this condition leads to the additional four equations

$$E_1^{(2)} = E_1^{(3)}, \quad E_2^{(3)} = E_2^{(4)}, \quad E_3^{(4)} = E_3^{(1)}, \quad E_4^{(1)} = E_4^{(2)} \quad (4.6)$$

when applied across the four boundaries meeting at the point 0. Thus our system of equations (4.2), (4.3), (4.4) and (4.6) is now over-determined unless

$$\sigma_1 + \sigma_3 = \sigma_2 + \sigma_4 \quad (4.7)$$

in which case four equations become redundant and the relation (4.5) still holds. The condition (4.7) includes the plane boundary ($\sigma_1 = \sigma_2$, $\sigma_3 = \sigma_4$, or $\sigma_1 = \sigma_4$, $\sigma_2 = \sigma_3$) as a special case, but excludes boundaries with corners. We conclude, therefore, that except for a few simple geometries it is not possible to formulate a finite difference representation of the general problem depicted in Fig. 1(b).

The way out of this dilemma is to look at equation (4.5) from a different point of view. It is identical in form to equation (4.1) with κ_0 replacing κ . Thus it may be regarded as the correct finite difference relation for a point in a region whose conductivity is the average of the conductivities of the surrounding regions. In other words, we are not really considering a region with sharp boundaries at all, but one in which the conductivity changes smoothly from one value to the next in a roughly linear manner. No boundary conditions have to be applied (there are no boundaries) and equation (4.5) can be used at each grid point once the appropriate value of κ_0 has been determined.

It is interesting to note that Lines & Jones (1973a,b) resorted to the same sort of interpretation when they replaced sharp boundaries by 'transition zones' in order to solve a three-dimensional induction problem. Rankin (1973) has claimed that their results were invalidated by this device but, on the contrary, this discussion has shown that the concept of the 'transition zone' is just as much implicit in the two-dimensional finite difference formulas, as it is in the three-dimensional ones. It is true, however, that the physical picture is more complicated in three-dimensions because of the fact that

$$\text{div } \mathbf{E} = -\mathbf{E} \cdot (\text{grad } \kappa) / \kappa \quad (4.8)$$

— by equation (2.2) — and this is non-vanishing when κ is a variable function of position, thereby indicating an accumulation of volume charge in the transition zones. This problem does not arise in two-dimensional E -polarization for which the right-hand side of equation (4.8) vanishes identically at all points.

B-POLARIZATION

We shall again begin our discussion by attempting to analyse the configuration of Fig. 1(b), this time for a B -polarization field. Since κ is regarded as constant in each of the four regions surrounding the point 0, equations (4.2) will continue to hold for the B -field. The value of B is uniquely defined at each grid point by virtue of the continuity of the tangential magnetic field. According to equation (2.6) the continuity of the tangential electric field across line A in Fig. 1(b) can be expressed by central difference formulas in the form

$$\rho_1(B_4^{(1)} - B_2) = \rho_4(B_4 - B_2^{(4)}), \quad \rho_2(B_4^{(2)} - B_2) = \rho_3(B_4 - B_2^{(3)}) \quad (4.9)$$

and across line B by

$$\rho_1(B_1 - B_3^{(1)}) = \rho_2(B_1^{(2)} - B_3), \quad \rho_4(B_1 - B_3^{(4)}) = \rho_3(B_1^{(3)} - B_3) \quad (4.10)$$

where we have substituted $\rho_i = 1/\kappa_i$ ($i = 1, 2, 3, 4$).

Once again the eight unknown fictitious values satisfy a system of eight equations whose matrix of coefficients has rank 7, and it is easily shown by elimination that they possess a solution if and only if

$$\rho_{41}B_1 + \rho_{12}B_2 + \rho_{23}B_3 + \rho_{34}B_4 = (4\rho_0 + ih^2)B_0 \quad (4.11)$$

where

$$\rho_{ij} = \frac{1}{2}(\rho_i + \rho_j), \quad \rho_0 = \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4). \quad (4.12)$$

The corresponding result obtained by Jones & Pascoe (1971) was not the same because they again used one-sided difference formulas for first derivatives and derived their final formula from an over-determined system of equations. In fact we have been unable to find a satisfactory interpretation of their B -polarization formula, except to note that it reduces to the correct form when $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$. Ward *et al.* (1973) on the other hand have used the central difference formulas to derive equations for two sub-cases of the general configuration in Fig. 1(b), namely the 'vertical discontinuity' ($\sigma_1 = \sigma_4, \sigma_2 = \sigma_3$) and the 'three-way corner' ($\sigma_1 = \sigma_4, \sigma_2 \neq \sigma_3$), which do agree with the corresponding special cases of our equation (4.11). But new problems of inconsistency appear in their analysis of the 'wedge-corner' ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma \neq \sigma_4$) shown in Fig. 1(c). This is because the five-point finite difference formula for the differential equation satisfied by B in the three-quarter space of conductivity σ is indistinguishable from equation (4.1) for the uniform region of Fig. 1(a). Thus Ward *et al.* obtained two inconsistent equations for the nodal point 0, one being equation (4.1) – containing no fictitious values – and the other resulting from the equation for the corner region with its two fictitious values (at nodes 2 and 3) eliminated by the boundary conditions on the tangential electric field. Their quoted final formula appears to have been found by adding the two inconsistent equations together, whereas the limiting form of equation (4.11) is obtained by adding the equations weighted in the ratio 3:1.

We have derived equation (4.11) by imposing the boundary conditions on the tangential electric and magnetic fields, just as we obtained equation (4.5) for the E -polarization field. However, since displacement currents have been neglected, we must also ensure that the normal component of the current density $\mathbf{j} = \sigma\mathbf{E}$ is continuous across the boundary between two conducting media. According to equation (2.6) we see that this corresponds to a statement of the natural boundary conditions that $\partial B/\partial y$ is continuous across line A and that $\partial B/\partial z$ is continuous across line B . Expressing these conditions in terms of fictitious values we obtain the additional four equations

$$B_1^{(2)} = B_1^{(3)}, \quad B_2^{(3)} = B_2^{(4)}, \quad B_3^{(4)} = B_3^{(1)}, \quad B_4^{(1)} = B_4^{(2)}. \quad (4.13)$$

Thus we have reached the same impasse as before. When all the boundary conditions are taken into account we obtain too many equations for the number of unknown fictitious values. Indeed it is immediately apparent that equations (4.13) are inconsistent with equations (4.9) and (4.10) unless

$$\sigma_1\sigma_3 = \sigma_2\sigma_4. \quad (4.14)$$

It is easy to see by a simple physical argument how this condition arises. Suppose that near the point 0 in the second quadrant the y -component of the electric field is E . By continuity of tangential electric fields the same component in the third quadrant must also be E . The normal component of current density approaching line B is therefore $\sigma_2 E$ in the second quadrant and $\sigma_3 E$ in the third, as shown in Fig. 1(d). Now this current is continuous across line B , so that the y -component of the electric field just to the right of the point 0 is

$\sigma_2 E/\sigma_1$ in the first quadrant, and $\sigma_3 E/\sigma_4$ in the fourth quadrant. These two fields are in turn continuous across line A and must, therefore, be equal to each other. If equation (4.14) is satisfied this will be so; otherwise the electric field must vanish or have a singularity at the node 0 which suggests that we are dealing with an improperly devised mathematical model of the physical situation we are trying to represent.

Even if the condition (4.14) is satisfied, six of the fictitious values can still be eliminated from equations (4.9), (4.10) and (4.13) leaving two unknowns satisfying the four equations (4.2). It can be shown that two of these remaining equations become redundant if either (i) $\sigma_1 = \sigma_2$ or (ii) $\sigma_2 = \sigma_3$. Taken together with (4.14), (i) implies $\sigma_3 = \sigma_4$ and (ii) implies $\sigma_4 = \sigma_1$. We conclude, therefore, that our derivation of equation (4.11) is generally valid only at vertical or horizontal plane boundaries.

At this point we are forced back to the idea of abandoning the concept of sharp boundaries in favour of a conductivity which is a smoothly-varying function of position, just as we did for E -polarization. In B -polarization, however, this is more than a simple re-interpretation of a given finite difference formula because it means the introduction of new terms into the basic differential equation, namely those in equation (2.8) whose coefficients involve derivatives of ρ . In the subsequent sections we describe a way of doing this and obtain the appropriate finite difference formulas for a mesh with variable node spacings. Apart from the fact that this new approach will remove the difficulties we have encountered in this discussion it is also a natural one for generalization to three-dimensional models.

5 The conductivity model

Let the yz -plane be covered by a mesh whose nodes (m, n) , $1 < m < M$, $1 < n < N$, correspond to the points $y = y_m$, $z = z_n$. In this notation, the left and right side boundaries and the top and bottom boundaries are at $y = y_1$, $y = y_M$, $z = z_1 < 0$ and $z = z_N > 0$ respectively. For simplicity we assume in this paper that $z = z_N$ is the surface of a superconductor. We take $z_q \equiv 0$ so that the nodes corresponding to $n = q$ are on the surface of the Earth. (In B -polarization problems the surface $z = 0$ is the upper boundary of the numerical grid, i.e. $q = 1$ in B -polarization.) Variable node spacings are introduced by the definitions

$$h_m = y_{m+1} - y_m, \quad k_n = z_{n+1} - z_n \quad (1 < m < M - 1, 1 < n < N - 1). \quad (5.1)$$

The numerical model is shown in Fig. 2.

In the region $z < 0$ the conductivity is zero everywhere, giving

$$\kappa_{m,n} = 0 \quad (1 < n < q - 1). \quad (5.2)$$

Otherwise the conductivity values are not specified at the nodes themselves, but at the centres of the rectangular elements of the mesh. We define $\sigma_{m+\frac{1}{2}, n+\frac{1}{2}}$ ($1 < m < M - 1$, $q < n < N - 1$) to be the given conductivity at $(m + \frac{1}{2}, n + \frac{1}{2})$, i.e. the point $y = y_m + \frac{1}{2}h_m$, $z = z_n + \frac{1}{2}k_n$, and we envisage a smooth variation in conductivity between neighbouring values. The actual form of this variation is not specified in the numerical model. Thus we are at liberty to assign to the nodes themselves any suitable conductivity values which are consistent with a reasonable functional behaviour between adjacent conductivities. For example, we could assume that either σ or the resistivity $1/\sigma$ changes linearly from one value to the next (with some rounding at the ends to form a smooth join). It turns out, however, that neither of these choices is satisfactory. In fact we shall find it appropriate to postulate two different types of variation, one for E -polarization and the other for B -polarization.

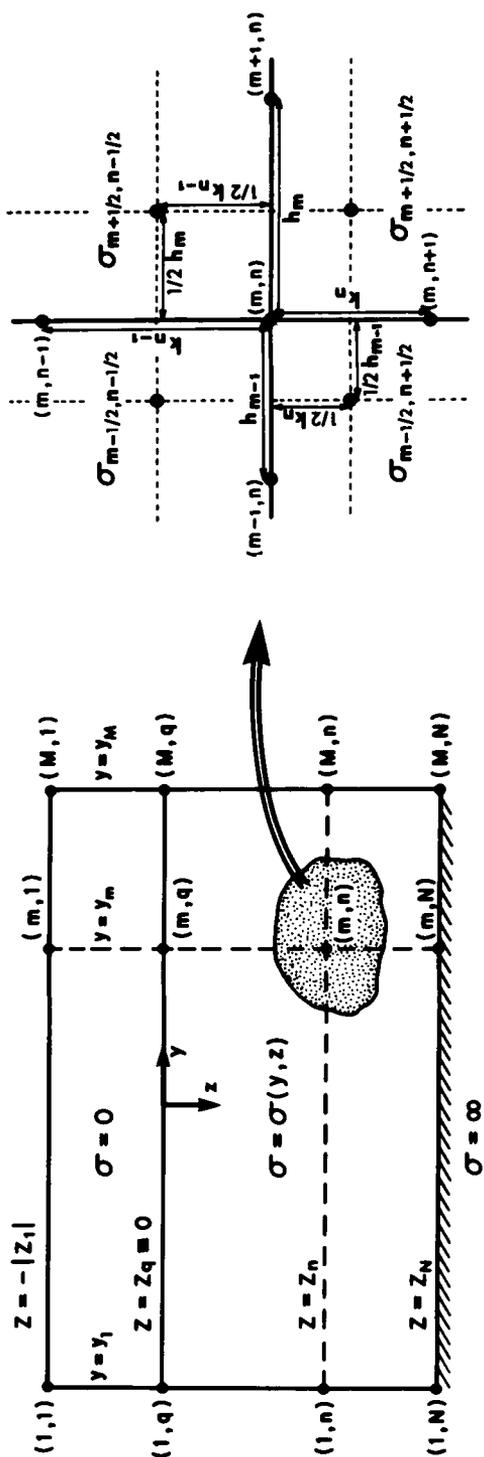


Figure 2. The numerical model.

E-POLARIZATION

Since we have assumed that $\partial\sigma/\partial y \rightarrow 0$ as $|y| \rightarrow \infty$ the conductivity values on, for example, the left side of the grid are defined by $\sigma_{1,n+\frac{1}{2}} = \sigma_{\frac{1}{2}n+\frac{1}{2}}$, ($q \leq n \leq N-1$). There is no difficulty in using fictitious values to derive the one-dimensional finite difference equation at a node $(1, n)$ because condition (4.7) is always satisfied at a boundary between two regions when the conductivity is only varying with depth. Thus we can regard the node $(1, n)$ as being on a *sharp* boundary between the region $z_{n-1} < z < z_n$ of constant conductivity $\sigma_{1,n-\frac{1}{2}}$ and the region $z_n < z < z_{n+1}$ of constant conductivity $\sigma_{1,n+\frac{1}{2}}$. Following the procedure described in Section 4, accounting this time for the different grid spacings when representing derivatives by central difference formulas, we can readily show that

$$\frac{E_{1,n+1}}{k_n(k_n+k_{n-1})} - \frac{E_{1,n}}{k_n k_{n-1}} + \frac{E_{1,n-1}}{k_{n-1}(k_n+k_{n-1})} = \frac{1}{2} i \kappa_{1,n} E_{1,n} \quad (5.3)$$

for $q \leq n \leq N-1$, where

$$\kappa_{1,n} = \frac{k_{n-1} \kappa_{1,n-\frac{1}{2}} + k_n \kappa_{1,n+\frac{1}{2}}}{k_n + k_{n-1}}. \quad (5.4)$$

Now equation (5.3) is precisely the finite difference representation of equation (3.1) at $(1, n)$ assuming that $\kappa^- = \kappa_{1,n}$ there. Thus the finite difference equation (5.3) holds equally well if the conductivity varies from one value to the next in such a way that the value of κ at a side node is an average of the two given values above and below weighted according to the formula (5.4). The numerical model is just not precise enough to distinguish between sharp boundaries and a variable conductivity of this type.

We now generalize this result to two dimensions. For consistency with one-dimensional equations we must average the conductivity in the same way in both the y - and z -directions. Thus we shall define

$$\kappa_{m,n} = \frac{K_{m,n} + K_{m,n-1} + K_{m-1,n} + K_{m-1,n-1}}{(h_m + h_{m-1})(k_n + k_{n-1})} \quad (5.5)$$

at the node (m, n) , $2 \leq m \leq M-1$, $q \leq n \leq N-1$, where

$$K_{m,n} = h_m k_n \kappa_{m+\frac{1}{2},n+\frac{1}{2}}.$$

By analogy with equation (5.4), at the nodes on the extreme right of the grid we have

$$\kappa_{M,n} = \frac{k_{n-1} \kappa_{M,n-\frac{1}{2}} + k_n \kappa_{M,n+\frac{1}{2}}}{k_n + k_{n-1}} \quad (5.6)$$

for $q \leq n \leq N-1$. Note that if the surface of the Earth is interpreted as a sharp boundary, then the conductivity at the node (m, q) must be taken as 0 on $z = -O$ and $\sigma_{m,q+\frac{1}{2}}$ on $z = +O$ where

$$\kappa_{m,q+\frac{1}{2}} = \frac{h_m \kappa_{m+\frac{1}{2},q+\frac{1}{2}} + h_{m-1} \kappa_{m-\frac{1}{2},q+\frac{1}{2}}}{h_m + h_{m-1}}. \quad (5.7)$$

However if the surface is regarded as a transition zone between $\sigma_{m,q+\frac{1}{2}}$ and $\sigma_{m,q-\frac{1}{2}} = 0$, (5.5) gives, for $2 \leq m \leq M-1$,

$$\kappa_{m,q} = \frac{k_q (h_m \kappa_{m+\frac{1}{2},q+\frac{1}{2}} + h_{m-1} \kappa_{m-\frac{1}{2},q+\frac{1}{2}})}{(h_m + h_{m-1})(k_q + k_{q-1})}. \quad (5.8)$$

B-POLARIZATION

As in the case of *E*-polarization we begin by considering a node $(1, n)$ on the left side of the mesh and assume that it is on a sharp boundary between the two regions $z_{n-1} < z < z_n$ and $z_n < z < z_{n+1}$ of constant conductivities $\sigma_{1, n-\frac{1}{2}}$ and $\sigma_{1, n+\frac{1}{2}}$ respectively. Since we may use the method of fictitious values at plane boundaries without difficulty, we can follow the procedure described in Section 4 to obtain the finite difference equation

$$\frac{\rho_{1, n+\frac{1}{2}} B_{1, n+1}}{k_n(k_n + k_{n-1})} + \frac{\rho_{1, n-\frac{1}{2}} B_{1, n-1}}{k_{n-1}(k_n + k_{n-1})} = \left(\frac{k_{n-1}\rho_{1, n+\frac{1}{2}} + k_n\rho_{1, n-\frac{1}{2}}}{k_n^2 k_{n-1} + k_n k_{n-1}^2} + \frac{i}{2} \right) B_{1, n} \tag{5.9}$$

for $2 \leq n \leq N - 1$. (Recall $q = 1$ for *B*-polarization.) After some algebraic rearrangement this equation can be put in the form

$$\begin{aligned} & \frac{2\rho_{1, n} B_{1, n+1}}{k_n(k_n + k_{n-1})} - \frac{2\rho_{1, n} B_{1, n}}{k_n k_{n-1}} + \frac{2\rho_{1, n} B_{1, n-1}}{k_{n-1}(k_n + k_{n-1})} \\ & + \left(\frac{\partial \rho}{\partial z} \right)_{1, n} \left(\frac{k_{n-1} B_{1, n+1}}{k_n(k_n + k_{n-1})} + \frac{(k_n - k_{n-1}) B_{1, n}}{k_n k_{n-1}} - \frac{k_n B_{1, n-1}}{k_{n-1}(k_n + k_{n-1})} \right) = i B_{1, n} \end{aligned} \tag{5.10}$$

where

$$\rho_{1, n} = \frac{k_n \rho_{1, n+\frac{1}{2}} + k_{n-1} \rho_{1, n-\frac{1}{2}}}{k_n + k_{n-1}} \tag{5.11}$$

$$\left(\frac{\partial \rho}{\partial z} \right)_{1, n} = \frac{\rho_{1, n+\frac{1}{2}} - \rho_{1, n-\frac{1}{2}}}{\frac{1}{2}(k_n + k_{n-1})}. \tag{5.12}$$

The interpretation of equation (5.10) is clear. It is a direct finite difference representation of equation (3.14) at the node $(1, n)$ under the assumption that $\rho^- = \rho_{1, n}$ and $\partial \rho^- / \partial z = (\partial \rho / \partial z)_{1, n}$ there. Thus equation (5.9) serves equally well if instead of a sharp boundary at $(1, n)$ we imagine that the conductivity is varying smoothly from $\sigma_{1, n-\frac{1}{2}}$ to $\sigma_{1, n+\frac{1}{2}}$ in such a way that ρ attains the weighted average value given by (5.11) at the node, and has a slope there given by (5.12). In fact this slope is just the linear gradient between the given values of ρ immediately above and below the node, and we know by the mean value theorem that this slope *must* be attained somewhere in the interval if σ (and hence ρ) is smoothly varying.

The generalization to two-dimensions is quite straightforward. At a node (m, n) , $2 \leq m < M - 1$, $2 \leq n \leq N - 1$, we define

$$\rho_{m, n} = \frac{R_{m, n} + R_{m, n-1} + R_{m-1, n} + R_{m-1, n-1}}{(h_m + h_{m-1})(k_n + k_{n-1})} \tag{5.13}$$

where $R_{m, n} = h_m k_n \rho_{m+\frac{1}{2}, n+\frac{1}{2}}$, and

$$\left(\frac{\partial \rho}{\partial y} \right)_{m, n} = \frac{2}{(h_m + h_{m-1})(k_n + k_{n-1})} \left(\frac{R_{m, n} + R_{m, n-1}}{h_m} - \frac{R_{m-1, n} + R_{m-1, n-1}}{h_{m-1}} \right) \tag{5.14}$$

$$\left(\frac{\partial \rho}{\partial z} \right)_{m, n} = \frac{2}{(h_m + h_{m-1})(k_n + k_{n-1})} \left(\frac{R_{m, n} + R_{m-1, n}}{k_n} - \frac{R_{m, n-1} + R_{m-1, n-1}}{k_{n-1}} \right). \tag{5.15}$$

In the last two formulas the slopes have simply been expressed as linear gradients between the appropriate weighted average values of ρ . For example, the right-hand side of equation (5.15) is equivalent to the gradient $(\rho_{m,n+\frac{1}{2}} - \rho_{m,n-\frac{1}{2}})/\frac{1}{2}(k_n + k_{n-1})$.

By analogy with equations (5.11) and (5.12), when $m = M$ we have

$$\rho_{M,n} = \frac{k_n \rho_{M,n+\frac{1}{2}} + k_{n-1} \rho_{M,n-\frac{1}{2}}}{k_n + k_{n-1}} \quad (5.16)$$

$$\left(\frac{\partial \rho}{\partial z}\right)_{M,n} = \frac{\rho_{M,n+\frac{1}{2}} - \rho_{M,n-\frac{1}{2}}}{\frac{1}{2}(k_n + k_{n-1})} \quad (5.17)$$

for $2 \leq n \leq N-1$.

In order to solve B -polarization problems it is also necessary to assign conductivities to the nodes $(m, 1)$ and (m, N) , $1 \leq m \leq M$, treated as lying on the surfaces $z=+0$ and $z=z_N-0$ respectively. In other words we regard the surfaces of the Earth and superconductor as sharp boundaries with nodes on their inside faces adjoining the finitely-conducting region $0 < z < z_N$. We now make the obvious assumption that the conductivity does not vary in the z direction from its specified values at depth $\frac{1}{2}k_i$ up to the surface $z=0$, and likewise that it is unchanging from its values at depth $z_N - \frac{1}{2}k_{N-1}$ down to the surface $z = z_N$. Thus, we have

$$\rho_{1,1} = \rho_{\frac{3}{2},\frac{3}{2}}, \quad \rho_{M,1} = \rho_{m-\frac{1}{2},\frac{3}{2}}, \quad \rho_{1,N} = \rho_{\frac{3}{2},N-\frac{1}{2}}, \quad \rho_{M,N} = \rho_{M-\frac{1}{2},N-\frac{1}{2}} \quad (5.18)$$

and for $2 \leq m \leq M-1$

$$\rho_{m,1} = \frac{h_m \rho_{m+\frac{1}{2},\frac{3}{2}} + h_{m-1} \rho_{m-\frac{1}{2},\frac{3}{2}}}{h_m + h_{m-1}}, \quad \rho_{m,N} = \frac{h_m \rho_{m+\frac{1}{2},N-\frac{1}{2}} + h_{m-1} \rho_{m-\frac{1}{2},N-\frac{1}{2}}}{h_m + h_{m-1}} \quad (5.19)$$

$$\left(\frac{\partial \rho}{\partial y}\right)_{m,1} = \frac{\rho_{m+\frac{1}{2},\frac{3}{2}} - \rho_{m-\frac{1}{2},\frac{3}{2}}}{\frac{1}{2}(h_m + h_{m-1})}, \quad \left(\frac{\partial \rho}{\partial y}\right)_{m,N} = \frac{\rho_{m+\frac{1}{2},N-\frac{1}{2}} - \rho_{m-\frac{1}{2},N-\frac{1}{2}}}{\frac{1}{2}(h_m + h_{m-1})} \quad (5.20)$$

and for $1 \leq m \leq M$

$$(\partial \rho / \partial z)_{m,1} = (\partial \rho / \partial z)_{m,N} = 0. \quad (5.21)$$

Finally, we shall find it convenient in later work to define the expressions

$$\begin{aligned} \gamma_{m,n} &= \rho_{m,n} + h_{m-1} (\partial \rho / \partial y)_{m,n}, & \delta_{m,n} &= \rho_{m,n} - h_m (\partial \rho / \partial y)_{m,n} \\ \zeta_{m,n} &= \rho_{m,n} + k_{n-1} (\partial \rho / \partial z)_{m,n}, & \eta_{m,n} &= \rho_{m,n} - k_n (\partial \rho / \partial z)_{m,n}. \end{aligned} \quad (5.22)$$

6 Finite difference equations for E -polarization

In order to obtain the boundary values round the edge of the grid it is first necessary to solve the one-dimensional equation (3.1) at $y = y_1$ and $y = y_M$, ($0 < z < z_N$) subject to the boundary condition (3.2). This can usually be done analytically but it is probably easier to solve numerically using the finite difference equation (5.3) on $y = y_1$ and the equivalent one on $y = y_M$. The boundary condition (3.2) can be translated into finite difference form by regarding the surface as a sharp boundary and expanding E in a Taylor series downwards

from $z = z_q$. Substituting for first and second derivatives from (3.2) and (3.1) and neglecting terms $O(k_q^3)$ we obtain (on $y = y_M$)

$$E_{M,q+1} + i\omega k_q H = (1 + \frac{1}{2}ik_q^2 \kappa_{M,q+\frac{1}{2}}) E_{M,q} \tag{6.1}$$

At $z = z_N$, the surface of a superconductor, we have

$$E_{M,N} = 0. \tag{6.2}$$

For $q + 1 \leq n \leq N - 1$, the equation equivalent to (5.3) is

$$k_{n-1} E_{M,n+1} + k_n E_{M,n-1} = \frac{1}{2}(k_n + k_{n-1})(2 + ik_n k_{n-1} \kappa_{M,n}) E_{M,n}. \tag{6.3}$$

The system (6.1), (6.2) and (6.3) which comprises $N - q + 1$ equations can be solved to give the $N - q + 1$ boundary values $E_{M,n}$ ($q < n \leq N$). A similar set of equations, obtained by formally replacing M by 1, gives the boundary values $E_{1,n}$ ($q < n \leq N$).

Across the entire bottom boundary $z = z_N$, which is the surface of a superconductor, we have

$$E_{m,N} = 0 \quad (1 \leq m \leq M). \tag{6.4}$$

The remaining boundary values are given by the formula (3.8). Written in the form

$$E_{m,n} = \left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{y_m}{|z_n|} \right) E_{1,q} + \left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{y_m}{|z_n|} \right) E_{M,q} + i\omega H |z_n| \tag{6.5}$$

this equation gives the boundary values on $y = y_1$ and $y = y_M$ ($1 \leq n \leq q - 1$) when $m = 1$ and $m = M$ respectively, and the values along the top boundary $z = z_1$, when $n = 1$.

It only remains to represent equation (2.7) by a finite difference equation at all the interior nodes of the mesh. For $2 \leq m \leq M - 1$ and $2 \leq n \leq N - 1$ we have

$$\begin{aligned} & \frac{2k_n}{h_m + h_{m-1}} \left(E_{m+1,n} + \frac{h_m}{h_{m-1}} E_{m-1,n} \right) + \frac{2h_m}{k_n + k_{n-1}} \left(E_{m,n+1} + \frac{k_n}{k_{n-1}} E_{m,n-1} \right) \\ & = \left(\frac{2k_n}{h_{m-1}} + \frac{2h_m}{k_{n-1}} + ih_m k_n \kappa_{m,n} \right) E_{m,n} \end{aligned} \tag{6.6}$$

where $\kappa_{m,n}$ is defined by (5.2) and (5.5). It is clear that this reduces to the form of equation (4.5) when $h_m = h_{m-1} = k_n = k_{n-1} = h$.

The corresponding equation derived by Jones & Pascoe (1971) differs from equation (6.6) in the last term where they have a simple average of the conductivities rather than the weighted average $\kappa_{m,n}$. This is because their procedure involved adding together the four equations (4.2), as explained in Section 4, and although it gave the correct result for equal grid spacings it fails to yield a properly weighted conductivity when the spacings are variable. This means that their general formula is inconsistent with the simpler formulas applicable at plane boundaries, for, as we have seen, the whole point of weighting the average of the conductivity was to preserve this consistency. Thus the use of their equation could lead to unnecessary inaccuracies at the surface of the Earth, and also prevent a proper matching of the solution with the side boundary conditions for a layered earth (Pascoe & Jones 1972), unless the perpendicularly directed grid spacings on either side of plane boundaries are always chosen to be equal.

The magnetic field components can be obtained from E by differentiation according to equation (2.5). For reasons to be discussed in Section 7 we prefer to compute them at the points $(m + \frac{1}{2}, n + \frac{1}{2})$ where the conductivity values themselves are specified, rather than at the nodes (m, n) . Using the standard four-point formulas for first derivatives we obtain

$$(B_y)_{m+\frac{1}{2}, n+\frac{1}{2}} = (E_{m+1, n} + E_{m, n} - E_{m+1, n+1} - E_{m, n+1})/2i\omega k_n \quad (6.7)$$

$$(B_z)_{m+\frac{1}{2}, n+\frac{1}{2}} = (E_{m+1, n+1} + E_{m+1, n} - E_{m, n+1} - E_{m, n})/2i\omega h_m \quad (6.8)$$

for $1 < m < M - 1$, $1 < n < N - 1$. The magnetic field at the surface of the Earth is of sufficient interest to warrant separate calculation. The vertical component is given at the points $(m + \frac{1}{2}, q)$, $1 < m < M - 1$, by the central difference formula

$$(B_z)_{m+\frac{1}{2}, q} = (E_{m+1, q} - E_{m, q})/i\omega h_m. \quad (6.9)$$

The horizontal component is more easily obtained at the nodes (m, q) . Regarding the surface as a sharp boundary we expand E in a Taylor series downwards from (m, q) and neglect terms $O(k_q^3)$ to get

$$E_{m, q+1} = E_{m, q} + k_q(\partial E/\partial z)_{m, q} - \frac{1}{2}k_q^2\{(\partial^2 E/\partial y^2)_{m, q} - i\kappa_{m, q} + \frac{1}{2}E_{m, q}\}$$

where the last term has been obtained from (2.7). A similar expression is obtained by expanding E upwards from (m, q) . Substituting for $\partial E/\partial z$ from (2.5) and eliminating $\partial^2 E/\partial y^2$ we obtain

$$(B_y)_{m, q} = \frac{ik_q k_{q-1}}{\omega(k_q + k_{q-1})} \left\{ \frac{E_{m, q+1}}{k_q^2} - \frac{E_{m, q-1}}{k_{q-1}^2} - \left(\frac{1}{k_q^2} - \frac{1}{k_{q-1}^2} + \frac{1}{2}i\kappa_{m, q+\frac{1}{2}} \right) E_{m, q} \right\}. \quad (6.10)$$

7 Finite difference equations for B -polarization

Along the upper boundary of the numerical grid, which is the surface of the Earth in this polarization, equation (2.12) gives

$$B_{m, 1} = H. \quad (7.1)$$

The lower boundary is the surface of the superconductor where we have $E_y = 0$ or, by equation (2.6), $\partial B/\partial z = 0$. Thus we can expand B in a Taylor series upwards from the nodes (m, N) , and neglect terms $O(k_N^3)$, to get

$$B_{m, N-1} = B_{m, N} + \frac{1}{2}k_N^2 \left\{ \frac{iB_{m, N}}{\rho_{m, N}} - \left(\frac{\partial^2 B}{\partial y^2} \right)_{m, N} - \left(\frac{1}{\rho} \frac{\partial \rho}{\partial y} \frac{\partial B}{\partial y} \right)_{m, N} \right\} \quad (7.2)$$

where we have substituted for $\partial^2 B/\partial z^2$ from equations (2.8) and (5.21), and where $\rho_{m, N}$ is defined by equations (5.18) and (5.19).

At the corners $(1, N)$ and (M, N) all y -derivatives must vanish. Thus at (M, N) , equation (7.2) gives

$$(\rho_{M, N} + \frac{1}{2}ik_N^2) B_{M, N} = B_{M, N-1}. \quad (7.3)$$

On the right side boundary $m = M$, $2 < n < N - 1$, the equation corresponding to (5.10) can be written as

$$k_{n-1}\zeta_{M,N}B_{M,n+1} + k_n\eta_{M,n}B_{M,n-1} = (k_n + k_{n-1})(\zeta_{M,n} + \eta_{M,n} + ik_nk_{n-1})B_{M,n} \quad (7.4)$$

where we have used the definitions (5.22). The $N - 1$ equations (7.3) and (7.4) can now be solved to determine the side boundary values in terms of $B_{M,1} = H$. A similar set of equations, obtained by formally setting $M = 1$, hold on the left side boundary.

The finite difference representation of equation (7.2) at the nodes (m, N) , $2 < m < M - 1$, along the lower boundary is

$$\begin{aligned} & \frac{\rho_{m,N} + \gamma_{m,N}}{h_m(h_m + h_{m-1})} B_{m+1,N} + \frac{\rho_{m,N} + \delta_{m,N}}{h_{m-1}(h_m + h_{m-1})} B_{m-1,N} + \frac{2\rho_{m,N}}{k_{N-1}^2} B_{m,N-1} \\ & = \left(\frac{2\rho_{m,N}}{k_{N-1}^2} + \frac{\gamma_{m,N} + \delta_{m,N}}{h_m h_{m-1}} + i \right) B_{m,N}. \end{aligned} \quad (7.5)$$

Finally the representation of equation (2.8) at the interior points of the mesh, $2 < m < M - 1$, $2 < n < N - 1$, is

$$\begin{aligned} & \frac{\rho_{m,n} + \gamma_{m,n}}{h_m(h_m + h_{m-1})} B_{m+1,n} + \frac{\rho_{m,n} + \delta_{m,n}}{h_{m-1}(h_m + h_{m-1})} B_{m-1,n} + \frac{\rho_{m,n} + \zeta_{m,n}}{k_n(k_n + k_{n-1})} B_{m,n+1} \\ & + \frac{\rho_{m,n} + \eta_{m,n}}{k_{n-1}(k_n + k_{n-1})} B_{m,n-1} = \left(\frac{\gamma_{m,n} + \delta_{m,n}}{h_m h_{m-1}} + \frac{\zeta_{m,n} + \eta_{m,n}}{k_n k_{n-1}} + i \right) B_{m,n}. \end{aligned} \quad (7.6)$$

This completes the solution of the B field.

The calculation of the electric field components, as given by equation (2.6), poses a slight difficulty. We have already remarked that at a node between different conductivity values the finite difference formulas for B are equally valid whether we regard the node as being in a region of smoothly varying conductivity, or on a plane boundary between two regions of (locally) different conductivity. This interchangeability of interpretation no longer applies to the electric field at the same node. Its normal component varies rapidly but continuously in a smooth transition zone, but is actually discontinuous, and therefore not uniquely defined, at a plane boundary. By simply replacing the magnetic field derivatives in equation (2.6) with central difference formulas we would obtain some sort of average electric field consistent with the former interpretation, but its value would be very sensitive to slight changes in position. We feel that more useful information is provided by an application of the central-difference formulas away from the regions of possible sharp variations in conductivity; for this reason we compute the electric field at points $(m + \frac{1}{2}, n + \frac{1}{2})$. Recalling that these are the points at which the actual conductivities of the model are defined and using the four-point formula for first derivatives, we obtain

$$(E_y)_{m+\frac{1}{2},n+\frac{1}{2}} = \frac{B_{m+1,n+1} + B_{m,n+1} - B_{m+1,n} - B_{m,n}}{2\mu_0 k_n \sigma_{m+\frac{1}{2},n+\frac{1}{2}}} \quad (7.7)$$

$$(E_z)_{m+\frac{1}{2},n+\frac{1}{2}} = \frac{B_{m,n+1} + B_{m,n} - B_{m+1,n+1} - B_{m+1,n}}{2\mu_0 h_m \sigma_{m+\frac{1}{2},n+\frac{1}{2}}} \quad (7.8)$$

for $1 < m < M - 1$, $1 < n < N - 1$.

Equation (7.7) does not give the horizontal electric field at the surface $z = 0$, which is the component of greatest interest in practical applications. It can be found by expanding B in a Taylor series downwards from the points $(m, 1)$, and expressing $\partial B/\partial z$ in terms of E_y by equation (2.6). According to equations (2.8), (5.21) and (7.1) we may also substitute iH/ρ for $\partial^2 B/\partial z^2$ at the surface, so that neglecting terms $O(k_1^3)$ we obtain

$$(E_y)_{m,1} = (\omega\rho_{m,1}/k_1)(B_{m,2} - H) - \frac{1}{2}i\omega k_1 H. \tag{7.9}$$

The reason for calculating the magnetic field components in Section 6 at the points $(m + \frac{1}{2}, n + \frac{1}{2})$ rather than at the nodes (m, n) was basically the same, but a little less obvious because in E -polarization the electric field is everywhere continuous since it is always tangential to conductivity boundaries. But σE is then discontinuous across a plane boundary and hence by equations (2.2) and (2.5) the normal gradient of the tangential magnetic field and the normal second derivative of the electric field are also discontinuous. It follows that at a node situated in the transition zone between different conductivities the gradient of the magnetic field and the second derivative of the electric field will be large, thereby causing the central difference formula to lose its order of accuracy at the node.

The finite-difference equations derived in this section do not agree with the corresponding B -polarization equations of Jones & Pascoe, even when the node spacings are equal. However, we have verified the correctness of our results (for both polarizations) by obtaining the self-same equations in a completely different way involving an integration of the differential equations over the elements of the mesh. The finite element method also leads to equations which are the same as ours within the order of accuracy of the finite difference approximation.

We conclude therefore that the B -polarization equations of Jones & Pascoe are incorrect except in regions of uniform conductivity. The extent to which their errors are important is examined in the next section where we consider a simple numerical example.

8 Numerical results

A numerical comparison has been made between the method given in this paper and the method of Jones & Pascoe (both with and without the correction of Williamson *et al.*). The model used (Fig. 3) consists of a uniform half-space, in which is embedded a square block of conductivity ten times as great. A grid of 41×41 points was used. The z -direction was vertical, increasing downwards, with the surface of the half-space at $z = 0$. The grid points were at the following values of z : $-7.0, -2.0, -0.5, -0.1, -0.025, 0.0, 0.025, 0.05, 0.1, 0.125, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.575, 0.60, 0.70, 0.80, 0.90, 1.0, 1.2, 1.4, 1.7, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0$. The

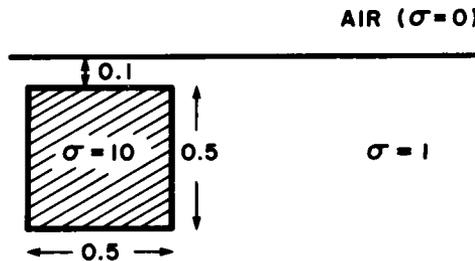


Figure 3. The conductivity model.

y -direction was horizontal, with $y = 0$ at the plane of symmetry. The grid points were at the following values of y : 0, 0.05, 0.10, 0.15, 0.20, 0.225, 0.25, 0.35, 0.50, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, together with the same negative values. The unit of distance is one skin-depth in the half-space. The conducting block is in the region $-0.25 < y < 0.25$, $0.1 < z < 0.6$. This model has been chosen to show the differences between the methods without unrealistic exaggeration; greater differences can easily be obtained by using a more extreme model. The vertical spacing on either side of the surface of the half-space has been made the same; much greater differences between the methods have been found in E -polarization when this is not so, as the equally weighted conductivity of Jones & Pascoe is then severely in error (*cf.* Section 6). The perpendicular grid spacings on either side of the faces of the conducting block were not the same.

The numerical equations were solved on an IBM 370 computer by a method of Gaussian elimination, using double-precision arithmetic throughout; the use of elimination avoids any problems with insufficient convergence of an iterative method. The computation was exactly the same for all methods, except for the few statements that calculate the coefficients in the numerical equations. The boundary conditions were exactly equivalent to those of Jones & Pascoe. The same model has also been run using Jones & Pascoe's (1971) published program, to check the results.

In E -polarization, Fig. 4(a)(b) show the real and imaginary parts of the electric field at the surface of the half-space, normalized so that the electric field is unity at the edge of the model. The most obvious feature is the difference between Jones & Pascoe's original formula and the others towards the edge of the model. This is due to their incorrect finite difference formulas for derivatives, which make the numerical solution incompatible with the analytic solution at the edge. This is overcome by the correction of Williamson *et al.* (1973), which makes the results closer to those obtained using the method of this paper. There remain differences of up to 3 per cent in the absolute magnitude of the field, which affect the imaginary part over a wide area of the model. A further run was made using the method of this paper on the same model, but with a square grid of side 0.05 in the central part of the model ($-0.5 \leq y \leq 0.5$, $-0.1 \leq z \leq 1.0$). The difference between this and the uneven grid used above was 0.3 per cent or less, which is much less than the discrepancy between our results and those of the other methods.

In B -polarization the magnetic field is necessarily unity across the surface, so Fig. 4(c)(d) show the horizontal electric field, calculated according to equation (7.7), and thus slightly below the surface. The extrapolation to the surface according to equation (7.9) has not been used here, to allow easier comparison with the results of Jones & Pascoe's program. Again the error of Jones & Pascoe's original formula towards the edge of the model is evident, and is overcome by the correction of Williamson *et al.* The other obvious feature is that the method of this paper gives a narrower dip over the conducting block. Although this can be considered as a small sideways shift of the curve, up to one grid position, it results in large differences (up to 30 per cent) between the field values calculated at fixed positions, because of the steepness of the curve. Examination of the results near the faces of the conducting block (not shown here) confirm this impression that Jones & Pascoe's method behaves as though the faces of the block were displaced outwards by up to one grid spacing from their true position. The use of an even grid over the central part of the model caused differences that were usually less than 1 per cent, but some differences of several per cent did occur on the edges, and particularly the corners, of the conducting block. But whenever the even grid gave a difference of more than 0.5 per cent, the modified method of Jones & Pascoe was different by at least five times as much. Thus the method of this paper is self-consistent to a considerably better degree than the differences between the methods.

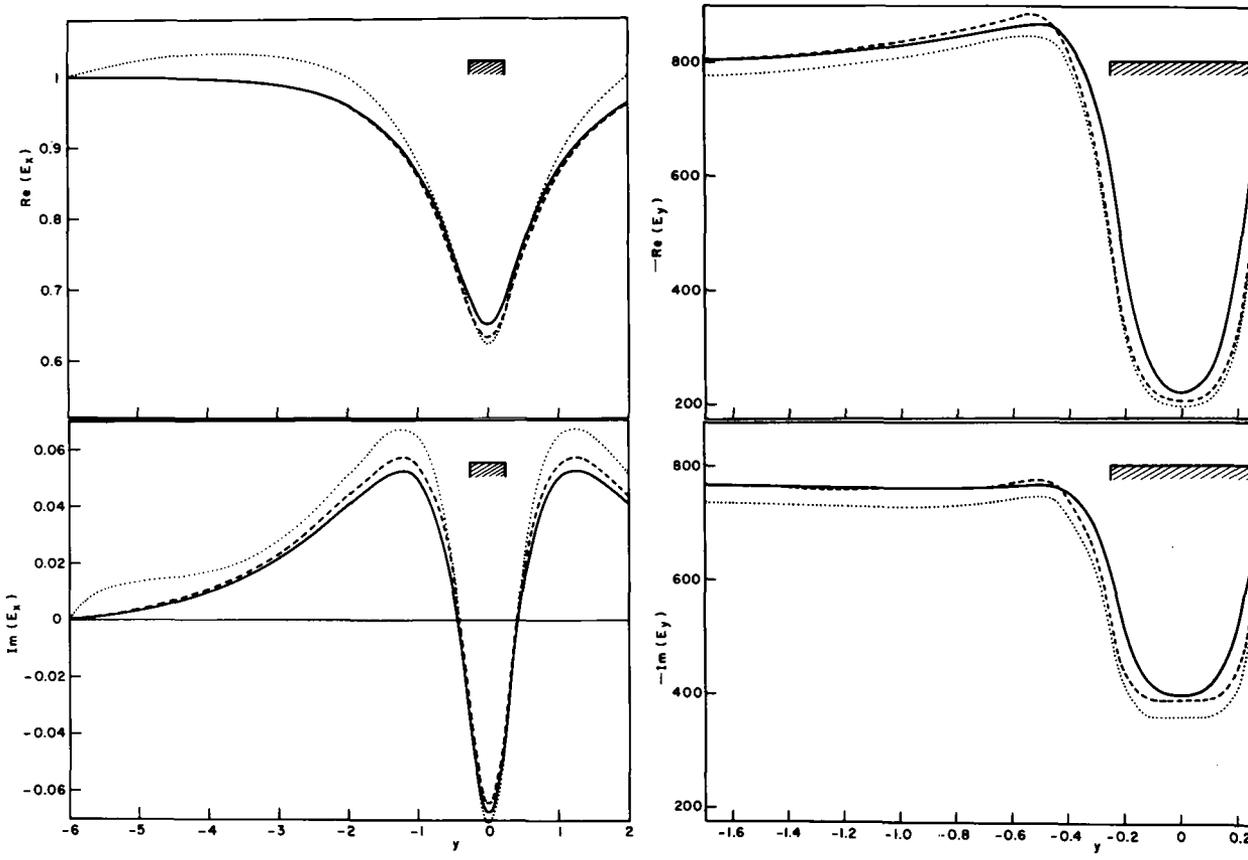


Figure 4. (a) Real, (b) Imaginary part of the electric field (E_x) at the surface of the half-space ($z = 0$) in E -polarization, (c) Real, (d) Imaginary part of the negative horizontal electric field ($-E_y$) just below the surface ($z = 0.0125$) in B -polarization. The solid lines (—) show the method of this paper. The dashed lines (---) show the method of Jones & Pascoe after correction by Williamson *et al.*; and the dotted lines (...) show the method of Jones & Pascoe as originally published. The shaded area shows the width and position of the conducting block.

9 Conclusions

The main conclusion to be drawn from this paper is that the finite-difference equations which have been used previously to solve two-dimensional problems of electromagnetic induction in the Earth need revising. For E -polarization the equations must be modified if the step sizes in the numerical grid are unequal across conductivity changes. For B -polarization the previously published equations are incorrect and can lead to substantial errors.

In order to establish the correct form of the finite-difference equations we have re-constructed the theory in a different way which gives a simple physical interpretation to the terms involving the conductivity in the relevant differential equations. The interpretation is chosen so that the finite-difference equations are consistent with those known to hold at plane boundaries and, in particular, in one-dimensional problems. While it might be argued that our method is more a well-motivated generalization of the one-dimensional equations rather than a rigorous development of the theory from first principles, we have confirmed that it is indeed correct by deriving exactly the same equations using more formal procedures, including the finite element method, and a method of integrating over the mesh elements. A detailed description of these and other methods, and a comparison between them, is clearly out of the question here. A comprehensive study of the whole subject will, however, be published as an internal report.

Integrating over the mesh elements is, in one sense, a more general method than the one we have described here because it does not depend on an assumed conductivity variation between the values specified in the model. (As a matter of fact, the particular variation assumed in B -polarization models is not unique. The average resistivity values at the nodes could have been weighted in any desired manner provided that the finite-difference formulas for the first derivatives of the field, considered as weighted averages of two one-sided derivatives, were modified accordingly.) An advantage of the approach used in this paper is that it has a simple physical interpretation which can be immediately generalized to three dimensions whereas the more formal development of the three-dimensional theory using the integration method requires rather more effort to obtain the equations. It is obvious what the appropriate generalizations of equations (5.5) and (5.13) to (5.15) should be to define $\kappa_{l,m,n}$, $\rho_{l,m,n}$ and $(\text{grad } \rho)_{l,m,n}$ respectively, where l denotes the node number in the x -direction. Thus a numerical solution of a three-dimensional problem could be obtained either in terms of the E -vector, in which case $\kappa_{l,m,n}$ would be used in the finite difference representation of equation (2.3), or in terms of the B -vector, in which case the definitions of $\rho_{l,m,n}$ and $(\text{grad } \rho)_{l,m,n}$ would be used in the finite-difference representation of equation (2.4).

In this connection, we should mention that Lines & Jones (1973a,b) developed their solutions in terms of the E -vector, but apparently did not use the properly averaged conductivity $\kappa_{l,m,n}$. We suspect, therefore, that their equations do not reduce to the correct form at plane boundaries when irregular mesh spacings are used. Their results, and those based on their equations which have been published in subsequent papers by Jones and co-workers, may require re-interpretation.

Acknowledgments

This work was supported by a National Research Council of Canada grant A3830, and by a University of Victoria Faculty Research Grant.

References

- Jones, F. W., 1973. Induction in a layered plane earth by uniform and non-uniform source fields, *Phys. Earth Planet. Int.*, 7, 282–293.

- Jones, F. W. & Pascoe, L. J., 1971. A general computer program to determine the perturbation of alternating electric currents in a two-dimensional model of a region of uniform conductivity with an embedded inhomogeneity, *Geophys. J. R. astr. Soc.*, **24**, 3–30.
- Jones, F. W. & Price, A. T., 1970. The perturbations of alternating geomagnetic fields by conductivity anomalies, *Geophys. J. R. astr. Soc.*, **20**, 317–334.
- Jones, F. W. & Thomson, D. J., 1974. A discussion of the finite difference method in computer modelling of electrical conductivity structures. A reply to the discussion by Williamson, Hewlett and Tammemagi, *Geophys. J. R. astr. Soc.*, **37**, 537–543.
- Kertz, W., 1954. Modelle für erdmagnetisch induzierte elektrische Ströme im Untergrund, *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. Abt. IIa*, 101–110.
- Lines, L. R. and Jones, F. W., 1973a. The perturbation of alternating geomagnetic fields by three-dimensional island structures, *Geophys. J. R. astr. Soc.*, **32**, 133–154.
- Lines, L. R. & Jones, F. W., 1973b. The perturbation of alternating geomagnetic fields by an island near a coastline, *Can. J. Earth Sci.*, **10**, 510–518.
- Pascoe, L. J. & Jones, F. W., 1972. Boundary conditions and calculation of surface values for the general two-dimensional electromagnetic induction problem, *Geophys. J. R. astr. Soc.*, **27**, 179–193.
- Raiche, A. P., 1974. An integral equation approach to three-dimensional modelling, *Geophys. J. R. astr. Soc.*, **36**, 363–376.
- Rankin, D., 1973. The perturbation of alternating geomagnetic fields by an island near a coastline: discussion, *Can. J. Earth Sci.*, **10**, 1702.
- Schmucker, U., 1971. Interpretation of induction anomalies above non-uniform surface layers, *Geophysics*, **36**, 156–165.
- Ward, S. H., Peeples, W. J. & Ryu, J., 1973. Analysis of geomagnetic data, *Methods in computational physics*, vol. 13, pp. 163–238, ed. Bruce A. Bolt, Academic Press, New York.
- Weidelt, P., 1975. Electromagnetic induction in three-dimensional structures, *J. Geophys.*, **41**, 85–109.
- Williamson, K., Hewlett, C. & Tammemagi, H. Y., 1974. Computer modelling of electrical conductivity structures, *Geophys. J. R. astr. Soc.*, **37**, 533–536.