THE MAGNETOTELLURIC INVERSE PROBLEM

ROBERT L. PARKER

Institute of Geophysics and Planetary Physics, Scripps Institution of Oceanography, University of California, San Diego, La Jolla, CA 92093, U.S.A.

Abstract. The magnetotelluric inverse problem is reviewed, addressing the following mathematical questions: (a) *Existence of solutions*: A satisfactory theory is now available to determine whether or not a given finite collection of response data is consistent with any one-dimensional conductivity profile. (b) *Uniqueness*: With practical data, consisting of a finite set of imprecise observations, infinitely many solutions exist if one does. (c) *Construction*: Several numerically stable procedures have been given which it can be proved will construct a conductivity profile in accord with incomplete data, whenever a solution exists. (d) *Inference*: No sound mathematical theory has yet been developed enabling us to draw firm, geophysically useful conclusions about the complete class of satisfactory models.

Examples illustrating these ideas are given, based in the main on the COPROD data series.

1. Introduction

This paper is a review of progress in the study of the magnetotelluric inverse problem for a very simple physical system – an electrically conducting halfspace in which the conductivity σ varies only with depth and is isotropic. Despite the simplicity of the mathematical model and of the governing differential equations, the relation between the observed quantities and the unknown conductivity profile is nonlinear; this fact makes the problem difficult and there are still some aspects of it that are far from understood. The usefulness of the model can best be judged by the large number of papers in the geophysical literature relying upon it for interpretational purposes and by the almost equally large number devoted to advancing the associated theory.

Let us first define the geophysical system more precisely and introduce some terminology. Electromagnetic induction takes place in the conducting halfspace driven by a periodic horizontal magnetic field, varying in time like $e^{i\omega t}$. The electric and magnetic fields found at the surface of the conductor are not independent of each other: the electric field is also horizontal and perpendicular to the magnetic field; its magnitude and phase relative to the magnetic field depend upon the conductivity below, which is of course the basis of the magnetotelluric method. For convenience we erect a Cartesian coordinate system with origin at the surface, z downward and x parallel to the magnetic field. Then using Weidelt's (1972) definition we introduce a complex response given by

$$c = -E_x/i\omega B_y.$$

This complex number varies with the frequency of the source field. Natural electromagnetic fields contain energy at all frequencies; if these fields are measured we may make estimates of c by Fourier analysis of the observed time series of E_x and B_y. Practical considerations limit these estimates to a finite collection of responses at a number of distinct frequencies:

Geophysical Surveys **6** (1983) 5-25. 0046-5763/83/0061-0005$03.15.

$$c_j = c(\omega_j), \quad j = 1, 2, \ldots N.$$

More conventionally in the geophysical literature observations are represented in terms of the apparent resistivity ρ_a and phase ϕ:

$$c = (\rho_a/\mu_0\omega)^{1/2}e^{i(\phi-\pi/2)}.$$

A uniform halfspace exhibits a constant value of ρ_a equal to the resistivity of the medium and Re c is twice the skin depth. When the conductivity varies with depth, this allows a rudimentary interpretation of ρ_a to be made as the approximate resistivity averaged over the penetration scale of the energy at the frequency in question. Inside the medium the electric field E (we drop the subscript x) obeys the differential equation

$$\frac{d^2E}{dz^2} = i\omega\mu_0\sigma(z)E. \tag{1}$$

A boundary condition is supplied to insure that no energy is provided to the system from below: we say $E \to 0$ as $z \to \infty$; alternatively the vertical extent of the conductor may be made finite and then $E = 0$ at the bottom. From Maxwell's equations it follows that

$$c = E(0)/\left.\frac{\mathrm{d}E}{\mathrm{d}z}\right|_0. \tag{2}$$

Our task then is to determine whatever we can about σ in (1) from the observations of c and its relation to E in (2). As I suggested in a general review (Parker, 1977a), inverse problems like this one raise a number of mathematical issues, which we consider in turn.

2. Existence of Solutions

We should first decide whether a given data set is compatible with the mathematical model. The simplifications of the model, such as the uniformity of the source field and the lack of lateral variations in σ, are such that we ought not be surprised when the measurements cannot be reconciled with any one-dimensional profile. The question of existence is whether there is any model at all which can adequately satisfy the observations. This raises the question of what is meant by agreement between the predictions of a model and the data. The easiest case to analyze is the idealized one in which the data are supposed to be exactly known: we have a set of complex numbers c_1, c_2, ..., c_N or, in this case precisely equivalently, pairs $(\rho_a, \phi)_1$, $(\rho_a, \phi)_2$, ..., $(\rho_a, \phi)_N$ corresponding to frequencies ω_1, ω_2, ..., ω_N. Weidelt (1972) gives 19 inequalities involving the real and imaginary parts of c, which every realizable data set must obey. (I should say at this point that Weidelt's paper, written over ten years ago, is a landmark in the study of the inverse magnetotelluric problem; it touches on all the important issues; it is rightly the most widely quoted paper in the literature and has had a profound influence on the subject.) Most of Weidelt's relations involve derivatives of c

which cannot be obtained exactly from discrete data; nonetheless, bounds on $dc/d\omega$ and $d^2c/d\omega^2$ can be obtained. The basis of Weidelt's inequality constraints is the fact that c must always be expressible as an integral over a nondecreasing real function $a(\lambda)$:

$$c(\omega) = \int_0^\infty \frac{da(\lambda)}{\lambda + i\omega}. \tag{3}$$

This is a Stieltjes integral, which for continuously increasing functions may be interpreted as

$$c(\omega) = \int_0^\infty \frac{da}{d\lambda} \frac{d\lambda}{\lambda + i\omega}.$$

However, for many conductivity profiles the function $a(\lambda)$ exhibits jumps, which contribute terms of the form

$$\frac{\Delta a_n}{\lambda_n + i\omega},$$

where $\Delta a_n > 0$ is the magnitude of the jump in a and λ_n is the value of λ where it is located. The function $a(\lambda)$ is called the spectral function for Equation (2) and plays a central role in the theory of that differential equation. The points λ at which $a(\lambda)$ increases correspond to eigenfrequencies $i\lambda$ of the system under the surface boundary condition $dE/dz = 0$.

Weidelt (1972) and several others (e.g. Bailey, 1970; Fischer and Schnegg, 1980) have discussed relationships between the real and imaginary parts of c and between ρ_a and ϕ. In a genuine response, these pairs of functions are not independent so that, for example:

$$\phi(\omega) = \frac{\pi}{4} - \frac{\omega}{\pi} \int_0^\infty \ln\left[\frac{\rho_a(\omega')}{\rho_a(\infty)}\right] \frac{d\omega'}{\omega'^2 - \omega^2}.$$

Such connections, called dispersion relations, result from the fact that $c(\omega)$ has no zeros or singularities below the real axis in the complex ω plane, when c is considered as a function of complex frequency. Unfortunately, such relations are weak constraints on the data for two reasons. First, they require knowledge of the response function for all ω, not just at $\omega_1, \omega_2, \ldots, \omega_N$; this is an unrealistic demand. Second, they are satisfied by any function that dies away fast enough at infinity and whose singularities and zeros lie above the real ω axis. This far less restrictive than Equation (3), which forces the zeros and singularities onto the positive imaginary ω axis.

The conditions discussed so far are only necessary conditions for a solution to exist for the one-dimensional inverse problem. This means there are functions satisfying them that are not valid responses. For example,

$$\tilde{c}(\omega) = \frac{1 - i}{\sqrt{\omega}} + \alpha \frac{1 + i\omega}{1 + i\omega - \omega^2}$$

satisfies the dispersion relations for any positive α; in addition, if $0 < \alpha < 0.048\,033$, $\tilde{c}$

satiesfies all Weidelt's inequalities as well. But $\tilde{c}$ has poles at $\omega = (1 \pm i\sqrt{3})/2$ which are inadmissible in a true one-dimensional response.

The author (Parker, 1980) has provided a complete theory for the existence of solutions based essentially on the representation (3). If the sampled responses are substituted into (3) we obtain

$$c_j = \int_0^\infty \frac{da(\lambda)}{\lambda + i\omega_j}, \quad j = 1, 2, \ldots, N, \tag{4}$$

where we may recall $a(\lambda)$ is a nondecreasing real function. These constraints can be viewed as requiring the consistency of a semi-infinite linear programming problem for the unknown a. I show that if there are any solutions to (4) at all there must be one in which $a(\lambda)$ consists of a function that is constant except at J points of discontinuity, where the function jumps and $J \leqslant 2N$. To realize the test practically requires the approximation of the integral by a sum, and the application of standard linear programming algorithms. The convergence of a sequence of matrix approximations to the true integral can be established mathematically and in many numerical tests performed by the author extremely rapid covergence is observed. If a given collection of data c_j passes the test then solutions σ do exist, provided we agree to admit perfectly thin conducting sheets of the type introduced by Price (1949); this is because the conductivities corresponding to spectral functions $a(\lambda)$ that increase at only a finite number of points consist of a series of delta functions:

$$\sigma(z) = \sum_{k=1}^{J} \tau_k \delta(z - z_k). \tag{5}$$

The class of models in this form is called D^+. Thus the test is a necessary and sufficient one for solutions to exist.

It may be disappointing that the nesessary and sufficient condition takes the form of a fairly complex calculation and cannot be summed up in a succinct formula. However, the condition is subtle, as the following example illustrates. Figure 1a shows a simple model made up of uniform layers, terminated by a uniform half-space; Figure 1b shows the corresponding complex response c and the fifteen frequencies at which the response is sampled for a test. (The model gives a reasonable fit to the COPROD data set and the frequencies are those appropriate to that series; we shall discuss the COPROD observations more fully in a moment.) I performed the following numerical experiment. Each of the numbers was rounded to four significant figures. The consistency of this slightly perturbed set was tested and I found that the conditions (4) could be satisfied only to about four figures. If the accuracy of the responses was improved the fit also improved, until at around seven significant figures no further improvement could be obtained; this plateau is probably a result of the finite accuracy of the computer arithmetic (some parts of the calculation are performed in single precision on a 32-bit computer). We may conclude that no one-dimensional model can exactly satisfy the responses that had been rounded to four figures. This and similar numerical experi-

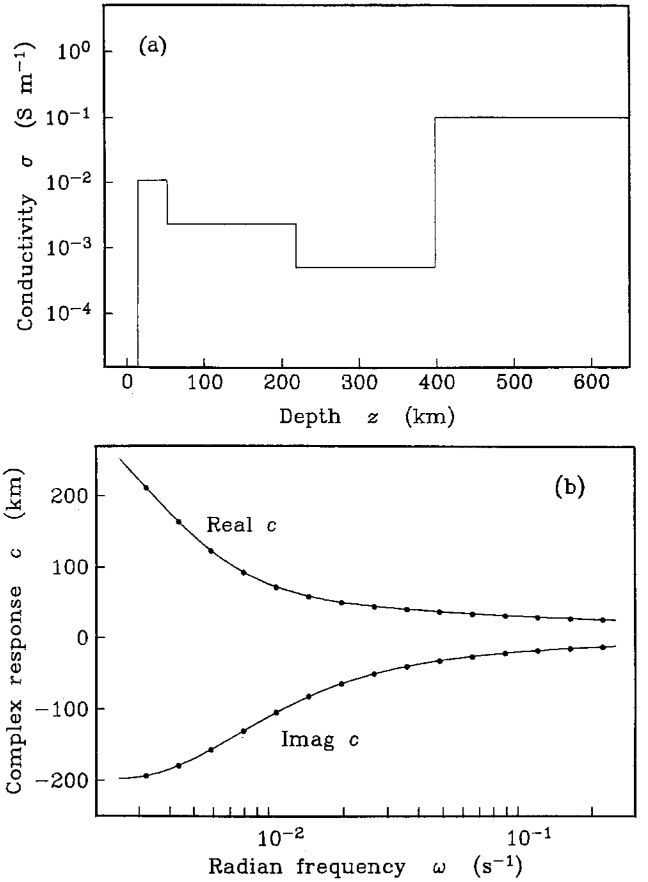

Fig. 1. (a) A simple uniform slab model. (b) Theoretical complex response c of the model; the fifteen sampling frequencies are the same as those in the COPROD data series.

Conductivity Model of Figure 1a

Conductivity (S/m)	Thickness (km)
0.0	13.426
0.010857	36.288
0.002275	167.145
0.00049519	179.28
0.1	infinity

Complex response c of Figure 1b

Radian freq (1/s)	Real c (km)	Imag c (km)
0.2205	25.759 254	−12.496 435
0.1632	27.239 246	−14.736 553
0.1205	28.896 709	−17.704 082
0.0891 2	30.874 535	−21.559 002
0.0657 9	33.299 438	−26.456 165
0.0486 7	36.165 070	−32.558 403
0.0359 7	39.518 440	−40.395 363
0.0265 9	43.619 400	−50.708 069
0.0196 5	49.232 292	−64.408 722
0.0145 2	57.726 242	−82.235 886
0.0107 4	71.072 678	−104.312 47
0.0079 35	91.884 109	−129.891 78
0.0058 66	122.336 62	−156.179 35
0.0043 35	163.057 92	−178.816 59
0.0032 04	211.345 70	−193.087 46

ments suggest the virtual impossibility of ever obtaining an *exact* fit to experimental data.

We are brought back to the question of what is meant precisely when we ask for a satisfactory agreement between theory and observation; clearly an exact fit is unreasonably demanding. The answer depends of course on the uncertainties ascribed to the data. It is important to remember that the actual quantities recorded are the horizontal components of electric and magnetic fields and that these time series must be processed to yield a magnetotelluric response. The uncertainties depend on the nature of the noise in the signals and on the way in which the time series are manipulated. If most of the noise is in one of the two signals (say the electric field is contaminated by random fields not due to telluric induction), then a standard theory (e.g. Bendat and Piersol, 1971) exists for optimal estimation of the transfer function between E and B; after its application one finds that at each frequency the response c consists of uncorrelated real and imaginary parts with equal variances, the value of the variance depending on the coherence between the signals. Furthermore, the probability function for $|c - \bar{c}|^2$ follows an F distribution curve with parameters that depend on the way in which the data are grouped. For reasonable choices of grouping the F distribution is well approximated by a Gaussian function. Unfortunately, it seems unlikely that the conditions for this theory apply to magnetotelluric observations and no entirely satisfactory theory has been developed as an alternative; in any event there is no generally agreed upon method for determining responses from field data. The situation is complicated by the fact that an impedance tensor must be estimated rather than a scalar. Bentley's (1973) study is widely quoted, but it should be noticed that his log-normal distribution for ρ_a was put forward entirely empirically. It seems to me that there is no good reason to believe that ρ_a and ϕ are the statistically appropriate pair of variables to use; in fact, we expect that the real and imaginary parts of the transfer function (and hence of c) will have a simpler statistical description than that of ρ_a and ϕ.

In the absence of a generally accepted theory for the statistics of the data, authors have felt free to pick their own criteria for acceptability of a fit, often without any theoretical justification whatever. Hobbs (1982) writes: 'MT analysts are notoriously optimistic in believing the significance of the errors accompanying their data, the result being that in some cases no model exists whose response fits . . .'; he uses this assertion as an argument for raising experimental error estimates if he is unable to find a suitable model! Fischer *et al.* (1981) adopt a measure of misfit consisting of the sum of squares of differences in ϕ and in $\ln \rho_a$, each quadratic term weighted by the inverse confidence interval (not the square of the interval as traditional practice would suggest). They also propose omitting a term form the sum whenever it is less than some arbitrary amount. Similar misfit measures have been used by many other authors (e.g. Larsen, 1981; Khachay, 1978; Fischer and Le Quang, 1981) presumably all motivated by Bentley's suggestion; weighting of the quadratic terms in the sum is often omitted. Another definition of a satisfactory model is given by Jones and Hutton (1979); they accept a conductivity profile (calling it 'acceptable at the 75% level'!) if more than 75% of the theoretical responses lie within the 95% confidence intervals of ρ_a and ϕ.

Most of the above treatments can be characterized as follows. The collection of measured responses may be represented as an element $D \in E^{2N}$, where E^{2N} is a normed $2N$-dimensional linear vector space; the predictions of the model are $\Theta \in E^{2N}$. The disagreement between D and Θ, the misfit, is measured by the norm of E^{2N}; if

$$\|\Theta - D\| < T,$$

where T is a tolerance, the model fits the data. The trouble with many of the above misfit measures is that there is no apparent rational basis for the choice of T that separates good models from bad ones. The following old fashioned approach seems quite reasonable. We set up the hypothesis that the true profile is the model under consideration and that the disagreement between model responses and observation are caused by the randomness in the data estimates. It is always possible to make some assessment of var D_j, the variance of the j-th observation; the crudest approach is to calculate the scatter obtained when the field time series are broken into a number of independent records and D_j is obtained from each. Then we can use a 2-norm in E^{2N}; let

$$\chi^2 = \|\Theta - D\|^2 = \sum_j (\Theta_j - D_j)^2/\text{var}\, D_j.$$

We must now calculate the probability that χ^2 would reach or exceed the observed level by chance; if the probability is too low (say less than 0.05) we must reject the model. When D_j are independent Gaussian random variances the probability is distributed as χ^2_n, or for large enough numbers of data, approximately normally. With more exotic distributions for D_j approximate confidence levels can be found by the central limit theorem. The number of 'degrees of freedom' is $2N$, the number of independent data in the sum. This treatment may be rough, particularly in view of the uncertainties in var D_j or the possible lack of statistical independence of the variables D_j, however the choice of T is founded in some sort of statistical model in contrast to most of those in the geophysical literature. The problem of existence is reduced to that of finding the model with χ^2 as small as possible; if this model is rejected so will every other model and we must conclude that there is no one-dimensional profile capable of reproducing the data. The criterion suggested here runs the risk of a type II statistical error (accepting the existence of a one-dimensional model when in fact none exists) because we have not accounted for the fact that the predictions Θ_j are not independent of the data. Nonetheless, I have found it to be far less generous than the *ad hoc* criteria currently fashionable.

With these ideas it is not hard to adapt the existence theory developed for exact data to the case of noisy data (Parker and Whaler, 1982). Now we minimize

$$\sum_{j=1}^{N} \left| c_j - \int_0^\infty \frac{\mathrm{d}a(\lambda)}{\lambda + i\omega_j} \right|^2 \frac{1}{s_j^2} \tag{6}$$

over all nondecreasing spectral functions $a(\lambda)$. This is a semi-infinite quadratic programming problem which can be solved by standard methods. It is found that, as with exact data, there is always a spectral function $a(\lambda)$ minimizing (6) which has only a

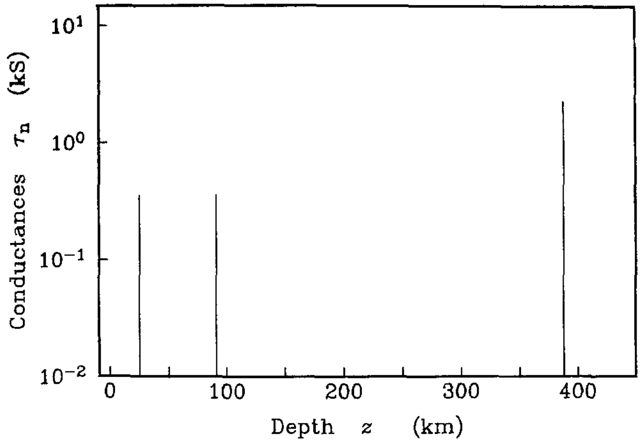

Fig. 2. Delta function solution with smallest misfit in the sense of χ^2 for the experimental COPROD data set (the fifteen well-estimated responses); $\chi^2_{min} = 33.7$.

Model of Figure 2

Depth (km)	Conductance (kS)
25.263	0.352 77
91.043	0.356 03
387.80	2.287 5

finite number of jumps and is otherwise constant. This means that (1) has only a finite number of eigenvalues, which can happen only if σ is in the form of (5), i.e. σ consists of a sum of delta functions. No other model can have a χ^2 smaller than the delta-function model that fits the data best.

As an illustration let us turn to the data set of the COPROD study organised by A. G. Jones. This set has been the subject of many investigations (e.g. Jones and Hutton, 1979; Larsen, 1981; Fischer and LeQuang, 1981; Hobbs, 1982; Parker, 1982). I followed the assumption that ρ_a and ϕ are uncorrelated and calculated an approximate standard deviation for them by dividing the given confidence interval by 3.92, the number appropriate for a Gaussian distribution. In (6) I included a modification to account for the covariance in c introduced by making ρ_a and ϕ uncorrelated variables with unequal variances. For the 15 well estimated responses, the minimum $\chi^2 = 33.7$, which is easily within the 95% value of 43.8; there is no doubt that one-dimensional profiles exist in accord with these data, because many investigators have already found them. The best fitting model has not been published before; it appears in Figure 2. When all 23 responses of the COPROD data set are included the smallest χ^2 increases to 1.66×10^4, which causes us to reject the possibility of a one-dimensional model with more than 99.9% certainty.

It is possible to minimize other misfit measures in (6) in place of the simple quadratic functional; then a nonlinear optimization scheme which applies the positivity constraints on $da(\lambda)$ must be used. As described in Section 4, Khachay (1978) has done something quite similar to this. Thus, through the use of the spectral function representation of a response, the problem of determining whether or not any one-dimensional models exist satisfying a practical data set has been solved. Best-fitting models are however not geophysically plausible because they consist of a series of delta-functions in conductivity.

3. Uniqueness

Suppose it has been decided that a given collection of response data is compatible with a one-dimensional conductivity profile; we may ask whether only one such profile fits the data, or whether more than one model can accomplish this feat. The answer depends on the type of data available. It has been long known (Tichonov, 1965; Bailey, 1970) that when the given responses are exact at all frequencies a solution, if it exists at all, is unique (at least, for a class of sufficiently smooth models). Obviously, since ρ_a and ϕ are inter-related, complete knowledge of ρ_a alone yields a unique solution too (as does the real part of c). Suppose c (or ρ_a) is known precisely for all ω with $\omega_1 < \omega < \omega_2$; since c is an analytic function of ω with singularities on the positive imaginary axis, it can be analytically continued into the complex ω plane and, more specifically, calculated everywhere on the real axis. Hence complete knowledge of c on any open interval is equivalent to knowledge of c at all frequencies, and therefore only one model can fit exact data given everywhere in a finite frequency band. Less well known perhaps is the fact that c (or ρ_a) can be unambiguously reconstructed from an infinite set of samples taken at evenly spaced frequencies: $\omega_0, \omega_0 + \Delta\omega, \omega_0 + 2\Delta\omega, \ldots$, for any $\omega_0, \Delta\omega > 0$ (see Lanczos Chapter 1, 1961). Again a solution is unique if its satisfies observations at all these frequencies.

Every one of the above cases requires knowledge of c or ρ_a at infinitely many frequencies, something which is, practically speaking, impossible. Because the unknown σ is a function, we should not expect it to be uniquely defined by a finite number of constraints even when these are given precisely. Certaintly we know from our discussion of existence that, if an ordinary (say piece-wise continuous) model satisfies the data, then there is another solution in terms of data functions. Figure 3 illustrates this; synthetic data consisting of the fifteen complex responses of Figure 1b are fitted very precisely (to about 4 parts in 10^6) by the delta function solution shown; recall that these data were generated from the model in Figure 1a. The profiles of Figures 1a and 3 are two completely different conductivities, both satisfying a finite data set essentially exactly. In fact it is possible to find certain finite data sets that can be satisfied exactly by only one model (Parker, 1980), but this situation must be regarded as anomalous.

The case of imprecise observations is trivial. If any model exists satisfying

$$\|\Theta - D\| < T,$$

then surely so do infinitely many others: the solution cannot be unique.

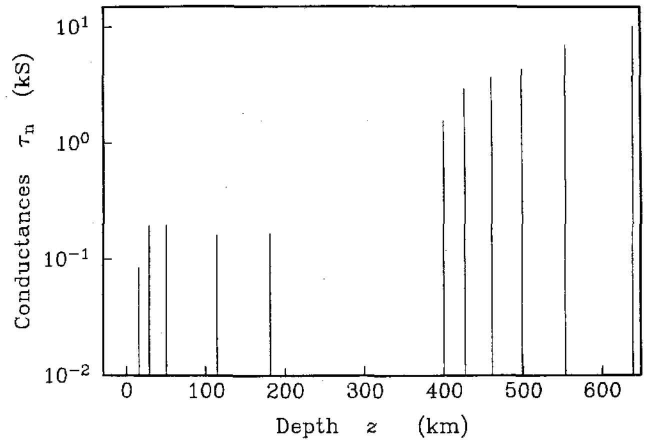

Fig. 3. A delta function model satisfying the synthetic responses of Figure 1b.

Model of Figure 3

Depth (km)	Conductance (kS)
16.075	0.085962
29.168	0.19570
50.269	0.19632
114.34	0.16127
181.56	0.16747
399.96	1.5747
426.62	2.9437
460.47	3.6631
408.51	4.3490
553.40	7.0210
638.96	10.094

4. Construction

We come now to the question that occupies that attention of the vast majority of geophysicists working on the magnetotelluric inverse problem: how can we find an example of a conductivity profile that fits our observations, provided of course that such a model exists? Until very recently there was no satisfactory existence theory and so the failure of a specific algorithm could be attributed to the inadequacy of the data rather than to any weakness in the modeling technique. Lack of uniqueness of the solutions means that there is a certain arbitrariness about the model obtained; it also sheds doubt on the usefulness of any individual profile for geophysical interpretation. Jones (1982) writes '... it is *axiomatic* in geophysical data interpretation to find the simplest model – or models – that satisfies the observed response ...' This widely held belief is surely reasonable, but it offers little help in suggesting a way out of the difficulty of arbitrariness because the idea of simplicity is quite subjective. In practice there are two schools of thought on the matter: one regards solutions consisting of a small number of uniform layers separated by discontinuities as simple models; the other elects smoothly varying functions with small gradients to be the preferred class. The discontinuous models may well be appropriate in the upper crust where large conductivity contrasts may be found at the contact between geologically different units; deeper in the Earth phase changes could cause sudden jumps in conductivity too, but it seems for some depth ranges more likely that smoothly increasing temperature will determine the behavior of σ and then a smooth function would seem to be the proper archetype.

Let us first review the discontinuous solutions, which have been the subject of intensive study (e.g. Jupp and Vozoff, 1975; Shoham *et al.*, 1978; Benvenuti and Guzzon, 1980; Larsen, 1981; Fischer and Le Quang, 1981). In these and other investigations the idea of simplicity is enforced by restricting the solutions to be in a class consisting of a relatively small number of homogeneous layers. Thus the set of unknown parameters (which may include layer thickness, but need not do so) can be considered to be a vector $p \in E^M$ where $M < 2N$. For this system the solution to the forward problem is easily written down; here we express it symbolically as the vector-valued function

$$\Theta : E^M \rightarrow E^{2N}$$

which gives the predictions of the model at the N frequencies. The ostensible objective is to find the simple model that fits the data best, where, for example, misfit is defined by a weighted 2-norm in E^{2N}. We are thus brought to the nonlinear minimization problem

$$\min_p \| \Theta(p) - D \|. \tag{7}$$

A favorite algorithm to perform this minimization is the Gauss-Newton interative scheme. At each vector p one linearizes the function Θ, representing it by two terms in its Taylor expansion:

$$\Theta(p + \Delta p) \approx \Theta(p) + \Delta p \cdot \nabla \Theta \tag{8}$$

from which a linear least squares system results for Δp. Ill-conditioning of this system may have to be brought under control by singular value decomposition (e.g. Jupp and Vozoff, 1975) or Marquardt-Levinson regularization (e.g. Benvenuti and Guzzon, 1980); also it is normal to find the next approximations in the sequence by sweeping through the vectors $p + \alpha \Delta p$ where $\alpha > 0$ [for a mathematical analysis see Luenberger (1973) and also Gill *et al.* (1982)]. Gauss-Newton is a descent method; if only local minima for (7) exist, the process must converge to one of them. Experience with algorithms of this kind has not been entirely satisfactory, however. With noisy data the true minimum may be at a point where conductivity or layer thickness is negative; explicit positivity constraints are easily introduced, although this appears never to have been done except by the device of modeling the logarithm of conductivity. More fundamental is the problem that if layer thickness is variable, the true minimum to (7) may not be achieved for any finite p. We know that the global minimum over all positive profiles occurs at a delta function model and thus $\|\Theta(p) - D\|$ will decrease indefinitely as $p \to \infty$ along some trajectory on which σ grows and the layer thickness diminishes. The number of layers need not be very large before this type of behavior is guaranteed: for example, with the COPROD data set (where $2N = 30$) we see from Figure 2 that a layered model with more than six layers of variable thickness must have its minimum misfit with p at infinity. One way out is to reduce the number of layers, but then it may be difficult to fit the data; the best-fitting model with five layers (found by Fischer and Le Quang, 1981) only just satisfies the COPROD data at the 95 % confidence level by the χ^2 test of Section 2. The other obvious remedy is to fix the layer thicknesses; now we may run into the problem that a fairly large number of layers may be required to get an acceptable fit and then simplicity has been sacrificed.

Several authors have given inversion schemes in terms of layers that do not rely on the minimization of the misfit, but recover the structure more or less directly (Nabetani and Rankin, 1969; Schmucker, 1974; Patella, 1976; Fischer *et al.*, 1981). In general terms, these methods usually exploit the fact that the high frequency response contains information about the shallow structure. The basic idea is to find the shallowest structure first, remove it and then proceed to the next level using lower-frequency data. These direct inversion procedures are reported by their authors to perform very well in practice. However, it has not been proved that they will always be able to construct a solution satisfying the observations, when it is known such solutions exist; the method to be described next does possess that virtue.

Recently Kathy Whaler and I (Parker, 1980; Parker and Whaler, 1981) have developed an inversion process yielding a layered model which is based upon the spectral function. We restrict the solution to a class called H^+ made up of models composed of uniform layers with

$$\mu_0 \sigma_n h_n^2 = d^2$$

where σ_n and h_n are the conductivity and thickness of the n-th layer, and d^2 is a prescribed constant. Loewenthal (1975a, b) first considered such models; they have the property that the attenuation factor of every layer is the same, and so Loewenthal calls them equal penetration models. For finite systems the response is a rational function of $\cosh d(i\omega)^{1/2}$ which allows a representation for c similar to (3) to be set up. Then for any fixed $d^2 \neq 0$, the best-fitting spectral function can be found by a quadratic program: the model itself is recovered by the construction of a continued fraction. When d^2 is small we find solutions consisting of thin, highly conductive zones separated by thick poorly conducting regions; indeed as $d^2 \to 0$, the solutions approach delta function models and χ^2 tends to its minimum possible value. Large values of d^2 yield layers that have a more even conductivity. Figure 4 shows how χ^2 varies with d^2 for the COPROD data set; also a number of solutions are shown. This construction technique has the advantage of being able to find models with misfits as close to $\chi^2_{\min}$ as desired; the solution is uniquely defined and can be calculated quite quickly.

We turn now to methods for building continuous conductivity profiles. Some of

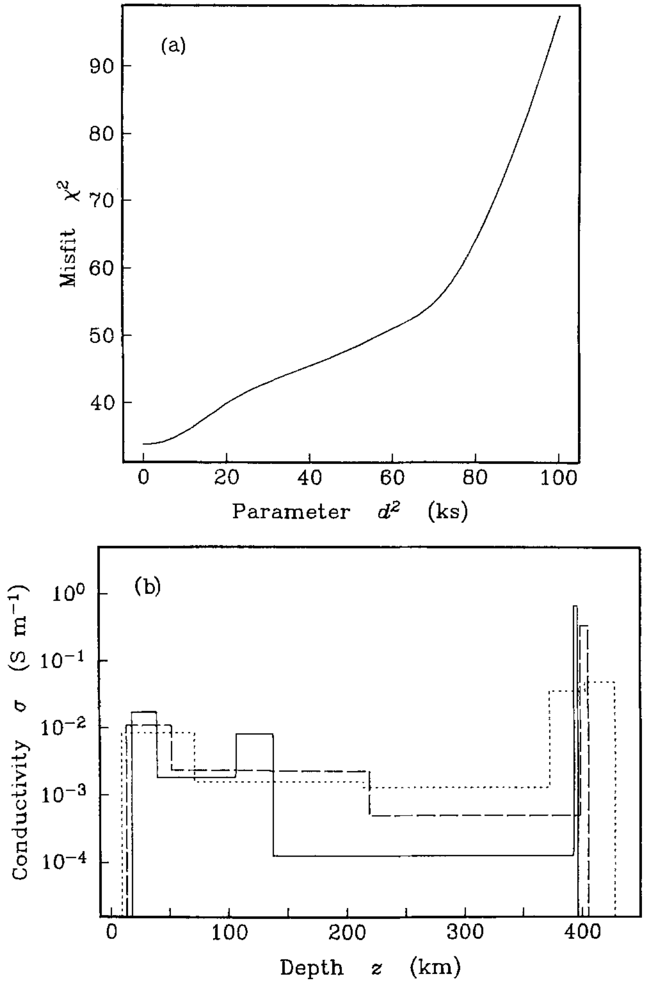

Fig. 4. (a) Misfit of the best-fitting solutions in H^+ as a function of the parameter d^2. (b) Some typical solutions; solid line $d^2 = 10$ ks, $\chi^2 = 35.6$; long dashed line $d^2 = 20$ ks, $\chi^2 = 39.8$; short dashed line $d^2 = 40$ ks, $\chi^2 = 45.4$.

Models of Figure 4b

$d^2 = 10$ ks		$d^2 = 20$ ks		$d^2 = 40$ ks	
Conductivity (S/m)	Thickness (km)	Conductivity (S/m)	Thickness (km)	Conductivity (S/m)	Thickness (km)
0.000 00	18.098	0.000 00	13.426	0.000 00	9.279 6
0.017 172	21.527	0.010 857	38.288	0.008 400 2	61.557
0.001 804 5	66.407	0.002 300 3	83.179	0.001 552 1	143.21
0.007 989 6	31.560	0.002 257 4	83.966	0.001 275 8	157.96
0.000 121 98	255.42	0.000 495 1	179.28	0.035 316	30.022
0.669 11	3.448 6	0.333 93	6.903 7	0.048 241	25.687

these can be seen as variants upon the original process described by Backus and Gilbert (1968). In its simplest form one performs the minimization (7), but now $p \in H$, where H is a space of functions, invariably some form of L^2. The gradient $\nabla\Theta$ becomes the Fréchet derivative, first given by Parker (1970) and later re-derived rigorously for the magnetotelluric problem (Parker, 1977b). It is amusing to note that, unmodified, the Gauss-Newton iteration can never converge to a satisfactory solution. This is easy to see: for any p, the linearized equations for Δp yield a family of solutions, each one of which claims to make $\|\Theta - D\|$ exactly zero; for practical data zero misfit cannot be achieved for positive p and therefore the iteration will move around forever or attempt to find negative components. In practice Gauss-Newton iteration is never used in its raw form: the solution vectors are unacceptably wiggly. Oldenburg (1979) uses a spectral expansion to select smooth components of the vector Δp; this is essentially the equivalent of singular value decomposition. One difficulty with this approach is that the particular solution obtained depends upon the starting guess; the process does not define a single result. Hobbs (1982) attempts to avoid the problem by finding models as close as he can to a uniform conductor: he introduces a bias into the data which pulls the responses towards those of a uniform model, Gauss-Newton iteration is used to improve the misfit, then the bias is reduced and the process repeated. Neither of these methods is certain to bring the misfit down to an acceptable level, although in actual application they appear to work quite well.

A very different approach is to use an analytical inversion scheme (e.g. Weidelt, 1972; Achache *et al.*, 1981). A major disadvantage, as Weidelt states, is that before practical data can be used, they must first be smoothly interpolated and extrapolated to produce a complete response function. Since the inverse problem is unstable (i.e. small changes in the response curve are not necessarily associated with small changes in the model), details of the data completion scheme can strongly influence the final solution. Furthermore, the conditions that insure the response curve really corresponds to any one-dimensional conductivity profile are very delicate (see Section 2) and few of the smoothing schemes are based on satisfactory functions; for example, the polynomial interpolations of Larsen (1975) and Hobbs (1982) are inconsistent with the existence conditions. Two smoothing schemes can guarantee success in this regard. One is that of Khachay (1978); he considers responses in the form

$$c(\omega) = \frac{2a_0}{\pi\sqrt{i\omega\mu_0\sigma_0}} \tan^{-1}\sqrt{\frac{i\omega}{\lambda_0}} + \sum_{k=1}^{L} \frac{a_k}{\lambda_k + i\omega} \tag{9}$$

choosing positive constants a_k, λ_k by a nonlinear program so as to match this function to the observed $\ln \rho_a$ and ϕ as well as possible in the least squares sense. Khachay uses (9) because it always yields a function c that can be generated from a proper spectral function $a(\lambda)$ in (3), a sufficient condition for obtaining true response functions. The other valid smoothing scheme is that of Parker and Whaler (1981); Khachay's method, slightly altered, is taken one step futher. We set

$$c(\omega) = \frac{1}{\sqrt{i\omega\mu_0\sigma_0}} + \sum_{k=1}^{L} \frac{a_k}{\lambda_k + i\omega} \tag{10}$$

and, with the χ^2 criterion of Section 2, we need use only a simple quadratic program to find the best fit. The advantage of (10) over (9) is that a very efficient analytic inversion via the Gel'fand-Levitan procedure can be performed; the integral equations arising from (10) possess degenerate kernels and so they may be expressed without approximations as finite matrix equations. The smooth models σ are said to belong to the class C^{2+}. The parameter σ_0 dictates the surface conductivity; it can be varied and this gives rise to a family of solutions. At one extreme, $\sigma_0 \to \infty$, we find very 'peaky' functions with misfits approaching $\chi^2_{\min}$; decreasing σ_0 leads to poorer misfit, but more nearly constant models. The COPROD data set has been analyzed in this way and the results appear in Figure 5. An annoying feature of these solutions is that the surface gradient $d\sigma/dz$ is always negative; work in progress promises to eliminate this nuisance.

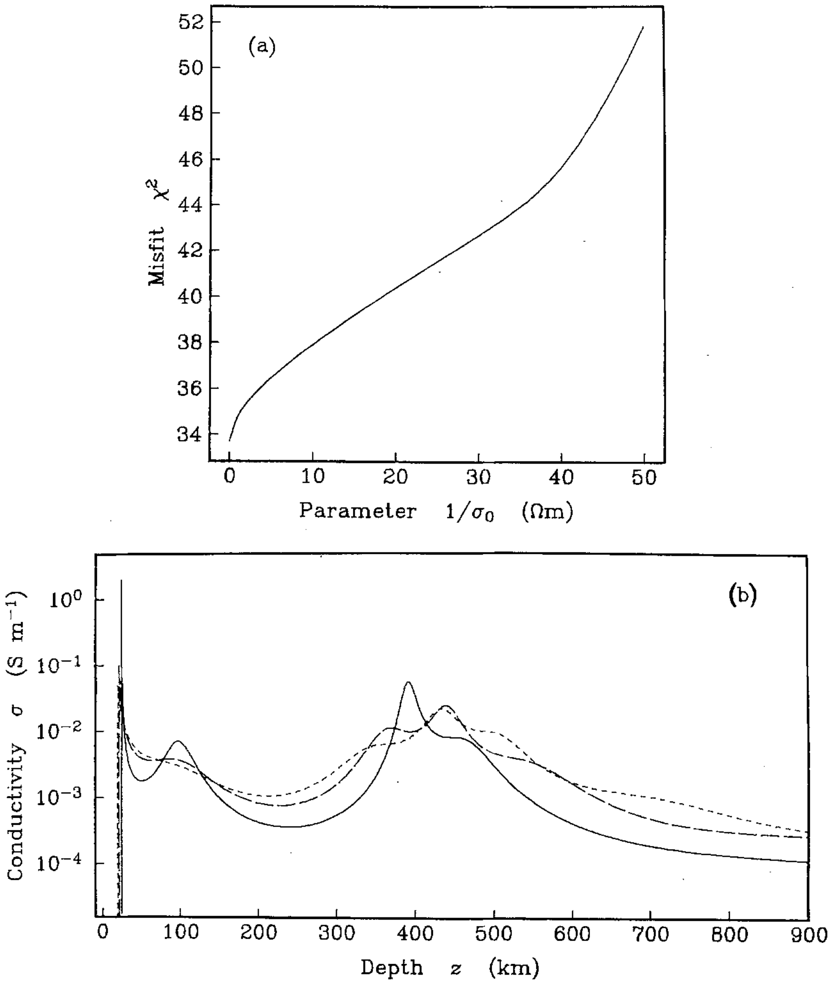

Fig. 5. (a) Misfit of the best-fitting solutions in the special class C^{2+} as a function of $1/\sigma_0$. (b) Some typical solutions; solid line, $\sigma_0 = 2$, $\chi^2 = 34.4$; long dashed line, $\sigma_0 = 0.1$, $\chi^2 = 37.8$; short dashed line, $\sigma_0 = 0.05$, $\chi^2 = 40.4$.

Models of Figure 5b

Depth (km)	σ (mS/m)	Depth (km)	σ (mS/m)	Depth (km)	σ (mS/m)	Depth (km)	σ (mS/m)
$\sigma_0 = 2.0$ S/m							
24.33	2000.00	90.17	6.273	362.8	4.411	507.8	2.425
25.44	42.00	100.8	6.885	368.0	6.539	527.7	1.403
25.59	35.71	110.6	4.910	373.4	10.82	544.4	0.9609
25.78	30.12	120.8	2.986	376.4	14.82	572.2	0.5876
26.00	25.22	131.5	1.851	379.7	21.70	620.7	0.3277
26.27	20.97	143.1	1.213	383.3	33.20	657.1	0.2443
26.59	17.82	157.7	0.8088	387.1	48.82	707.8	0.1840
26.98	14.22	181.9	0.5182	391.6	57.08	765.2	0.1492
28.74	7.691	221.2	0.3708	397.8	40.55	799.2	0.1364
31.93	4.167	272.2	0.4008	402.9	26.48	846.6	0.1238
37.60	2.456	304.6	0.5780	411.7	14.82	901.1	0.1130
46.92	1.802	322.5	0.8260	427.2	9.095	958.1	0.1032
60.09	1.936	337.5	1.273	451.1	8.159	1008.0	0.09483
72.70	2.899	348.1	1.922	478.7	5.810		
81.06	4.254	356.7	2.990	492.7	3.870		
$\sigma_0 = 0.1$							
21.19	100.0	136.4	1.977	436.1	24.87	938.9	0.2617
21.52	79.41	155.7	1.379	450.5	18.98	994.7	0.2383
21.72	70.35	187.1	0.9194	459.8	12.72		
21.95	61.83	234.3	0.7791	471.6	8.171		
22.23	53.92	274.3	1.035	491.4	5.229		
22.55	46.64	300.2	1.627	522.9	4.053		
22.93	40.03	314.2	2.339	561.2	2.997		
23.39	34.09	329.2	3.848	592.3	1.865		
25.37	20.19	339.5	5.712	625.0	1.111		
28.84	11.48	348.4	7.933	652.2	0.7787		
34.96	6.635	358.4	10.49	696.3	0.5113		
45.38	4.330	370.5	11.54	744.3	0.3834		
61.50	3.628	386.9	10.26	783.2	0.3325		
83.39	3.817	409.3	11.61	833.1	0.2983		
112.8	3.113	422.2	16.82	885.2	0.2789		
$\sigma_0 = 0.05$							
19.62	50.00	47.49	5.005	360.7	6.597	576.2	2.295
20.27	41.68	63.57	3.953	394.8	8.466	612.6	1.562
21.06	34.45	87.05	3.343	410.3	12.71	663.9	1.223
22.02	28.26	122.5	2.275	420.6	17.95	721.2	0.9544
23.19	23.02	151.4	1.563	432.4	22.48	756.9	0.7641
24.61	18.66	198.2	1.101	448.2	17.47	810.1	0.5302
25.46	16.74	252.5	1.300	474.3	10.81	845.2	0.4271
26.44	14.93	284.0	1.991	514.7	8.240	896.0	0.3343
30.50	10.33	308.2	3.258	536.0	5.141	954.2	0.2795
37.04	7.040	333.1	5.368	553.8	3.417	988.3	0.2633

Interpolation by cubic splines in the log of conductivity yields an accurate smooth curve.

Now that there is a theory for deciding whether or not a given data set is compatible with a one-dimensional profile, it seems reasonable to demand that, whenever solutions are known to exist, any satisfactory construction algorithm will always find one. Very few methods currently available can meet this requirement, although it is to be hoped the number will grow.

5. Inference

The most important task of inverse theory is to establish what conclusions can be legitimately drawn from the observations. The interpretation of profiles derived from response measurements is particularly risky in view of the inherent instability of the inverse problem. It would be accurate to characterize efforts on this difficult problem as very primitive at present.

Backus and Gilbert (1970) introduced the idea of resolution into geophysics; this is a length scale smaller than which details of the model cannot be perceived using the data in hand. Application of this intuitively appealing notion to nonlinear problems requires an approximation equivalent to the acceptance of (8) as an exact equation. Then the data would enable us to determine uniquely (aside from statistical scatter) certain averages in the form

$$A[\sigma; z_0] = \int \tilde{\delta}(z, z_0)\sigma(z)\mathrm{d}z, \tag{11}$$

where $\tilde{\delta}(z, z_0)$ is a peaked function with its maximum near z_0 and a width of approximately r, the resolution of the solution at the depth z_0. These ideas have been applied to electromagnetic inverse problems by a number of authors (e.g. Parker, 1970; Oldenburg, 1979, 1981; Larsen, 1981). As expected, the resolution deteriorates rapidly with depth. The linearization approximation is also at the heart of attempts to assess the uncertainty in the parameters governing the simple uniform layer models, or of extracting significant combinations of parameters (Jupp and Vozoff, 1975). While these results are very suggestive and undoubtedly useful in a general way, they cannot be regarded as constituting a mathematically sound solution to the problem of inference. Even without engaging in any analysis we can see that the linear approximation is unlikely to be accurate because the range of conductivities found in the solutions is so great; in fact, it will be shown that linearization can never be correct in this problem.

To avoid linearization some authors have employed the Monte Carlo method (e.g. Jones and Hutton, 1979; Jones, 1982; Connerney *et al.*, 1980). Here a very large number of profiles in a special class is generated at random; each one is tested to see if it satisfies the observations; if it does, the model is saved. A population of solutions is thus accumulated which, if numerous enough, covers the range of variability encompassed by the set of all possible solutions. The cardinal advantage of the Monte Carlo process is that no approximation is made in arriving at the population of solutions; the grave drawback is the extreme difficulty in generating an adequately large set of solutions. To reduce expense, the class of profiles generated must be severely restricted; in the case of Jones and Hutton (1979) for example, only three-layer uniform models were considered for most of their investigations. Also the criterion for acceptability of a profile as a solution may have to be made very loose; indeed, for the COPROD data set, every three-layer model is incompatible with the data at a probability of well over 95% according to the χ^2 criterion of Section 2. Data with much tighter error estimates than those of the COPROD series are not uncommon, and for these sets it will be prohibitively expensive to generate a large population of acceptable solutions by a random search.

I have recently obtained a negative result concerning possible inferences obtainable form practical response data (Parker, 1982). It is shown that models satisfying the data exist in which the conductivity below a critical depth is entirely arbitrary. There is therefore no information in the response data about σ below that depth; for the COPROD study the zone of total ignorance begins at about $z = 360$ km. From this we can show that linearization cannot be even approximately valid for the magnetotelluric problem. If it were, the value of A in (11) would be (approximately) the same for all solutions satisfying the data. But the arbitrariness of σ below some depth shows that A can be made to have any value at all, provided $\tilde{\delta}$ does not vanish identically in this region; it is easily shown that $\tilde{\delta}$ does not necessarily vanish there. Similarly, we are able to find two models σ_1 and σ_2 such that $\|\sigma_2 - \sigma_1\|$ (their distance apart in the model space) is arbitrarily large. These examples show that linearization can never be an adequate approximation.

In mathematical terms the inference problem can be restated as the search for the common properties shared by all models fitting the observations. Thus in Backus-Gilbert theory the value of A in (11) is the same for all valid solutions when the problem is linear and $\tilde{\delta}$ is specified in a certain way (it is a linear combination of Fréchet derivatives). It is frequently assumed that all the satisfactory profiles must lie within certain limits:

$$\sigma^- \leqslant \sigma \leqslant \sigma^+,$$

where σ^- and σ^+ are functions of depth which we could determine from the observations. The possibility of delta-function models, like those of Figures 2 and 3, illustrates what can easily be proved: there is no upper limit σ^+, and σ^- is zero for every depth. A potentially useful alternative is suggested by the idea of resolution: we consider the *average* value of σ in some fixed depth interval. In a nonlinear problem we could not expect this number to be determined exactly, but it might lie in a definite and perhaps interesting range. In the COPROD solutions illustrating this paper we see a definite hint of conductivity decrease in the top 200 km; perhaps it is the case that $\bar{\sigma}_1$, the mean conductivity in the range 0–100 km, is more than $\bar{\sigma}_2$, that is the next 100 km interval. Using only the profiles shown we find $4.8 \leqslant \bar{\sigma}_1 \leqslant 7.1$ and $0 \leqslant \bar{\sigma}_2 \leqslant 2.9$, in units of mS m^{-1}. This rather superficial test supports the idea that there is a conductivity decrease in the upper 200 km. One way of pursuing this idea is to extremize the linear functional

$$\int_{z_1}^{z_2} \sigma \, dz$$

subject to the constraints that $\sigma \geqslant 0$ and that the corresponding responses adequately fit the data. This is a semi-infinite nonlinear optimization problem; no rigorous theory exists for its solution as fas as I know.

The mathematical problem of establishing properties of the deep conductivity profile using practical magnetotelluric responses remains very incomplete. Progress in answering some of the other questions has been encouraging, however. It is to be hoped that geophysicists will focus their energies now onto the problem of making mathematically defensible inferences from the data.

6. Summary

The problem of existence of solutions has been essentially solved. If the observations are in the form of real and imaginary parts of a response or impedance, a quadratic program can find the smallest misfit possible between measured responses and any theoretical one-dimensional profile. There is no difficulty in principle in allowing the minimization of misfit in other variables too (like ln ρ_a and ϕ) although the author finds the reasons for wanting to do this far from compelling.

For practical data the matter of uniqueness of solutions is trivial: infinitely many profiles can fit the data if one can.

There has been some progress in the popular pastime of building models to fit the data. A few algorithms have been devised which it can be proved will converge to a solution provided one exists. Such methods must be favored over the majority of others where human intervention is required in the form of starting guesses, parameter adjustment or data deletion. Future work on one-dimensional algorithms should be expected to yield more methods with such guaranteed performance. It is hard to believe there is a need for any further effort on iterative inversion in terms of a small number of homogeneous layers.

Much work still needs to be done to develop a satisfactory method for making useful inferences about the conductivity based upon response measurements. Linearization is known to be an unreliable approximation, and the Monte Carlo approach is limited at best. The replacement of these techniques with a fully rigorous mathematical theory represents a considerable challenge for the future.

Acknowledgments

The author wishes to thank the National Science Foundation for its support of this work under Contract NSF EAR-79-19467.

References

Achache, J., Le Mouël, J. L., and Courtillot, V.: 1981, Long-Period Geomagnetic Variations and Mantle Conductivity: An Inversion Using Bailey's Method, *Geophys. J. Roy. Astr. Soc.* **65**, 579–601.

Backus, G. and Gilbert, F.: 1967, 'Numerical Applications of a Formalism for Geophysical Inverse Problems', *Geophys. J. Roy. Astr. Soc.* **13**, 247–276.

Backus, G. and Gilbert, F.: 1968, 'The Resolving Power of Gross Earth Data', *Geophys. J. Roy, Astr. Soc.* **16**, 169–205.

Bailey, R. C.: 1970, 'Inversion of the Geomagnetic Induction Problem', *Proc. Roy. Soc.* **315**, 185–194.

Bendat, J. S. and Piersol, A. G.: 1971, *Random Data: Analysis and Measurement Procedures*', John Wiley, New York.

Bentley, C. R.: 1973, 'Error Estimation in Two-Dimensional Magnetotelluric Analyses', *Phys. Earth. Planet. Interiors* **7**, 423–430.

Benvenuti, G. and Guzzon, M.: 1980, 'Inversion Method for Magnetotelluric Data', *Bull. Geof. Teor. App.* **85**, 47–58.

Connerney, J. E. P., Nekut, A., and Kuckes, A. F.: 1980, 'Deep Crustal Electrical Conductivity in the Adirondacks', *J. Geophys. Res.* **85**, 2604–2614.

Fischer, G. and Schnegg, P.-A.: 1980, 'The Dispersion Relations of the Magnetotelluric Response and Incidence in the Inverse Problem', *Geophys. J. Roy. Astr. Soc.* **62**, 661–673.

Fischer, G., Schnegg, P.-A., Peguiron, M. and Le Quang, B. V.: 1981, 'An Analytic One-Dimensional Magnetotelluric Inversion Scheme'. *Geophys. J. Roy. Astr. Soc.* **67**, 257–278.

Fischer, G. and Le Quang, B. V.: 1981, 'Topography and Minimization of the Standard Deviation in One-Dimensional Magnetotelluric Modelling', *Geophys. Phys. J. Roy. Astr. Soc.* **67**, 279–292.

Gill, P. E., Murray, W., and Wright, M. H.: 1981, *Practical Optimization*, Academic Press, New York.

Hobbs, B. A.: 1982, 'Automatic Model for Finding the One-Dimensional Magnetotelluric Problem', *Geophys. J. Roy. Astr. Soc.* **68**, 253–264.

Jones, A. G.: 1982, 'On the Electrical Crust-Mantle Structure in Fennoscandia: no Moho, and the Asthenosphere Revealed?, *Geophys. J. Roy. Astr. Soc.* **68**, 371–388.

Jones, A. G. and Hutton, R.: 1979, 'A Multi-station Magnetotelluric Study in Southern Scotland-II', *Geophys. J. Roy. Astr. Soc.* **56**, 351–368.

Jupp, D. L. and Vozoff, K.: 1975, 'Stable Iterative Methods for Inversion of Geophysical Data', *Geophys. J. Roy. Astr. Soc.* **42**, 957–976.

Khachay, O. A.: 1978, 'On Solving the Inverse Problem of Magnetotelluric Sounding for a Complex Impedance', *Phys. Solid. Ear., Izvestiya* **14**, 896–900.

Lanczos, C.: 1961, *Linear Differential Operators*, van Nostrand, New York.

Larsen, J. C.: 1975, 'Low Frequency (0.1 – 6.0 cpd) Electromagnetic Study of Deep Mantle Electrical Conductivity Beneath the Hawaiian Islands', *Geophys. J. Roy. Astr. Soc.* **43**, 17–46.

Larsen, J. C.: 1981, A New Technique for Layered Earth Magnetotelluric Inversion', *Geophysics* **46**, 1247–1257.

Luenberger, D. G.: 1973, *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, Mass.

Loewenthal, D.: 1975a, 'Theoretical Uniqueness of the Magnetotelluric Inverse Problem for Equal Penetration Discretizable Models', *Geophys. J. Roy. Astr. Soc.* **43**, 897–903.

Loewenthal, D.: 1975b, 'On the Phase Constraint of the Magnetotelluric Impedance', *Geophysics* **40**, 325–330.

Nabetani, S. and Rankin, D.: 1969, 'An Inverse Method of Magnetotelluric Analysis for a Multilayered Earth', *Geophysics* **34**, 75–86.

Oldenburg, D. W.: 1979, 'One-Dimensional Inversion of Natural Source Magnetotelluric Observations', *Geophysics* **44**, 1218–1244.

Oldenburg, D. W.: 1981, 'Conductivity Structure of the Oceanic Upper Mantle Beneath the Pacific Plate', *Geophys. J. Roy. Astr. Soc.* **65**, 359–394.

Parker, R. L.: 1970, 'The Inverse Problem of Electrical Conductivity in the Mantle', *Geophys. J. Roy. Astr. Soc.* **22**, 121–138.

Parker, R. L.: 1977a, 'Understanding Inverse Theory', *Ann. Rev. Earth, Plan. Phys.* **5**, 35–64.

Parker, R. L.: 1977b, 'The Fréchet Derivative for the One-Dimensional Electromagnetic Induction Problem', *Geophys. J. Roy. Astr. Soc.* **49**, 543–547.

Parker, R. L.: 1980, 'The Inverse Problem of Electromagnetic Induction: Existence and Construction of Solutions Based on Incomplete Data, *J. Geophys. Res.* **85**, 4421–4425.

Parker, R. L. 1982, 'The existence of a Region Innaccessible to Magnetotelluric Sounding', *Geophys. J. Roy. Soc.* **68**, 165–170.

Parker, R. L. and Whaler, K. A.: 1981, 'Numerical Methods for Establishing Solutions to the Inverse Problem of Electromagnetic Induction', *J. Geophys. Res.* **86**, 9574–9584.

Patella, D.: 1976, 'Interpretation of Magnetotelluric Resistivity and Phase Soundings over Horizontal Layers', *Geophysics* **41**, 96–105.

Price, A. T.: 1949, 'The Induction of Electric Currents in Nonuniform Sheets and Shells, *Quart. J. Appl. Math.* **2**, 283–310.

Schmucker, U.: 1974, 'Erdmagnetische Tiefensondierung mit langperiodischen Variationen, Protokoll über das Kolloquium 'Erdmagnetische Tiefensondierung', in Grafath-Bayern (ed.), A. Berktold, pp. 313–342.

Shoham, Y., Ginzburg, A. and Abramovici, F.: 1978, Structure in Central Israel from the Inversion of Magnetotelluric Data', *J. Geophys. Res.* **83**, 4431–4440.

Tichonov, A. N.: 1965, 'Mathematical Basis of the Theory of Electromagnetic Soundings', *USSR Comp. Math. & Phys.* **5**, 201–211.

Weidelt, P.: 1972, 'The Inverse Problem of Geomagnetic Induction', *Z. Geophys.* **38**, 257–289.