CAUCHY INTEGRAL ANALOGUES FOR THE SEPARATION AND CONTINUATION OF ELECTROMAGNETIC FIELDS WITHIN CONDUCTING MATTER

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Abstract. The main results in the theory of the interpretation of geopotential fields are generalized to the case of arbitrary variable electromagnetic fields by means of elaborating electrodynamic analogues for the integral of the Cauchy type.

The generalized Kertz method for separating a variable electromagnetic field into parts related to the sources located in different regions of space is elaborated on the basis of this technique. The generalized Kertz method allows the selection of external and internal, normal and anomalous parts of the geomagnetic field, as well as the separation of geomagnetic anomalies into the surface and deep components caused by conductivity inhomogeneities in the Earth's crust and upper mantle.

The theory of analytical continuation of variable electromagnetic fields in a conducting medium is also developed in the present work using the technique of analogues for the integral of the Cauchy type. It is shown that analytical continuation of a field downwards permits the determination of the location and form of deep geoelectric inhomogeneities according to the configuration of the isolines of flux functions for magnetic and electric fields.

1. Introduction

Elaboration of effective methods for the interpretation of electromagnetic anomalies caused by inhomogeneities in the structure of the Earth's crust and upper mantle is the challenge in today's geoelectric methods. Solution to this interesting problem meets a variety of difficulties related to the necessity to divide beforehand the fields into the external and internal, normal and anomalous, surface and deep parts, as well as to the determination of the parameters of deep geoelectric inhomogeneities. Various approaches to the solution of these questions have been discussed in the following publications: Rikitake, 1966; Schmucker, 1970; Rokityansky, 1972, and in numerous other papers. However, there is a long way to go to finish the problem. If the degree of perfection of the methods for solving the inverse problem in geoelectric sciences could be compared with that, for example, in gravimetry and magnetometry, then, without any doubts, the comparison would be in favour of the latter. At the same time, there are a variety of problems in the geoelectric sciences which can be successfully solved using principles analogous to those well developed in gravimetry and magnetometry. The problem concerns the separation of anomalous fields and the determination of the geometry of the bodies forming the anomalies. The majority of the results in the theory of potential (gravitational and static magnetic) fields have been obtained using the technique of the integral of the Cauchy type for complex-analytical functions. In the present review

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these results are generalized to variable electromagnetic fields by elaborating certain analogues for the integral of the Cauchy type.

2. Electrodynamic Analogues for the Integral of the Cauchy Type

It is worth remembering, first of all, how the concept of the integral of the Cauchy type is introduced in the theory of the functions of a complex variable. For functions of a complex variable the Cauchy integral formula is known, according to which from the values of the analytical function $f(\zeta)$ at the boundary C of a region D it is possible to determine $f(z)$ everywhere in D:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} \; ; \; z \in D \tag{1}$$

where z denotes any internal point of the region D confined by the contour C. If, on the contrary, the point z lies outside the region D, then according to Cauchy's theorem:

$$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} = 0 \; ; \; z \in D \tag{2}$$

($\bar{D} = D + C$ is the region D with the boundary C). The Cauchy integral (1) gives a reprentation of the function $f(z)$, analytical in the region D, through its boundary values. However, this integral will make sense also in the case when an arbitrary contour C and a certain continuous function $\varphi(\zeta)$:

$$K(z) = \frac{1}{2\pi i} \oint_C \frac{\varphi(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \; ; \tag{3}$$

on it are given in the complex plane.

The integral (3) is called the integral of the Cauchy type, and the function $\varphi(\zeta)$ is its density. The function $K(z)$ defined by the integral of the Cauchy type has a number of remarkable features, the basics of which are the following

(1) $K(z)$ is analytic at any point z which does not lie on the contour C.

(2) If $\varphi(\zeta)$ are the boundary values on C of the function $\varphi(z)$,analytic everywhere in D, then $K(z) \equiv \varphi(z)$ within D and $K(z) \equiv 0$ outside $\bar{D}$.

(3) Limiting values of the function $K(z)$ exist when z tends to a point on C from within and from outside the region D, however these limiting values are different, since a jump takes place when passing across C; the value of the jump is equal to the density $\varphi(\zeta)$ of the integral of the Cauchy type (the well-known formulas of Sokhotsky–Plemelj).

In the theory of two-dimensional potential fields the integrals of the Cauchy type are of exceptional importance. With the help of these, methods of the separation of fields (Kertz, 1954) and those of analytical continuation (Strakhov, 1972) are elaborated, problems concerning the ambiguity in solving inverse problems (Tsirulsky, 1969) are investigated and the location and form of the field sources (Strakhov, 1970) are determined.

We shall demonstrate that a theory similar to some extent to the theory of the integral of the Cauchy type for complex-analytical functions can also be developed for variable monochromatic quasi-steady electromagnetic fields.

Let S be a smooth closed surface confining the region D in the space. Introduce the notations

$$F^s(r^q, G, U) = \frac{-1}{4\pi} \iint_S \left\{ (n, U)\mathrm{grad}_\mu G + \left[[n \times U] \times \mathrm{grad}_\mu G \right] \right\} \mathrm{d}S^\mu ;$$

$$P^s(r^q, G, U) = \frac{-1}{4\pi} \iint_S [n \times U] G \mathrm{d}S^\mu ; \tag{4}$$

where $\boldsymbol{n}$ is the unit vector of the normal external to S. $G = G(r^q, r^\mu)$ is the fundamental Green's function for the Helmholtz equation:

$$\Delta G(r^q, r^\mu) = K^2 G(r^q, r^\mu) - 4\pi\delta(r^q - r^\mu) ; \tag{5}$$

(δ is the Dirac function), i.e.

$$G(r^q, r^\mu) = \frac{1}{|r^q - r^\mu|} \cdot \exp(-K|r^q - r^\mu|).$$

We call the electrodynamic analogues for the integral of the Cauchy type the following expressions:

$$\begin{aligned} H(r^q) &= F^s(r^q, G, U) + \sigma P^s(r^q, G, V) ; \\ E(r^q) &= F^s(r^q, G, V) + i\omega\mu P^s(r^q, G, U) ; \end{aligned} \tag{6}$$

where ω, μ, σ are certain positive constants subject to the condition: $-i\omega\mu\sigma = K^2$; $\boldsymbol{U}$, $\boldsymbol{V}$ are vector functions defined on S called the densities of the integrals of the Cauchy type and related to each other by:

$$(\boldsymbol{U}, \boldsymbol{n}) = \frac{i}{\omega\mu} \mathrm{div}_s [\boldsymbol{n} \times \boldsymbol{V}]$$

$$(\boldsymbol{V}, \boldsymbol{n}) = -\frac{1}{\sigma} \mathrm{div}_s [\boldsymbol{n} \times \boldsymbol{U}]$$

Here div_S is the symbol for the surface divergence.

The principal features of integrals (6) allowing us to call these analogues for the integral of the Cauchy type, are the following:

(1) Everywhere in the space, except on the surface S, expressions (6) describe the functions satisfying Maxwell's equations for a quasi-steady monochromatic field; assuming a time factor $\exp(-i\omega t)$:

$$\begin{aligned} \operatorname{curl} \boldsymbol{H} &= \sigma \boldsymbol{E} \\ \operatorname{curl} \boldsymbol{E} &= i\omega\mu \boldsymbol{H} \end{aligned} \tag{7}$$

Therefore, the constant σ and μ introduced earlier are identified with the electrical conductivity and magnetic permeability for a homogeneous conducting medium, and ω is identified with the circular frequency of field oscillations.

(2) If densities (6) are the limiting values on S of the functions $\boldsymbol{h}$ and $\boldsymbol{e}$ satisfying over the region D confined by S Maxwell's equations

$$\begin{aligned} \operatorname{curl} \boldsymbol{h} &= \sigma \boldsymbol{e} \ , \\ \operatorname{curl} \boldsymbol{e} &= i\omega\mu \boldsymbol{h} \ , \end{aligned} \tag{8}$$

then outside S analogues for the integral of the Cauchy type are equal to zero, and within S are equal to $\boldsymbol{h}$ and $\boldsymbol{e}$:

$$\boldsymbol{H}(r^q) = \begin{cases} 0 \ ; q \notin D \\ \boldsymbol{h}(r^q); q \in D \end{cases}; \quad \boldsymbol{E}(r^q) = \begin{cases} 0 \ ; q \notin \bar{D}; \\ \boldsymbol{e}(r^q); q \in D; \end{cases} \tag{9}$$

If the fundamental Green's function G is substituted in expressions (6) by an arbitrary solution g of the Helmholtz equation in D, then

$$\begin{aligned} &\boldsymbol{F}^S(r^q, g, \boldsymbol{h}) + \sigma \boldsymbol{P}^S(r^q, g, \boldsymbol{e}) \equiv 0, \\ &\boldsymbol{F}^S(r^q, g, \boldsymbol{e}) + i\omega\mu \boldsymbol{P}^S(r^q, g, \boldsymbol{h}) \equiv 0, \end{aligned} \tag{10}$$

where

$$\Delta g(r^q) = K^2 g(r^q) \ ; q \in D.$$

(3) For the limiting values of electrodynamic analogues for the integral of the Cauchy type at the surface of integration S, formulas similar to the conventional Sokhotsky–Plemelj formulas in the theory of the integral of the Cauchy type are valid, namely

$$\boldsymbol{H}^+(r^0) = \lim_{\substack{q \to 0 \\ q \in D}} \boldsymbol{H}(r^q) = \boldsymbol{F}^S(r^0, G, \boldsymbol{U}) + \sigma \boldsymbol{P}^S(r^0, G, \boldsymbol{V}) + \frac{1}{2} \boldsymbol{U}(r^0) \tag{11}$$

$$H^-(r^0) = \lim_{\substack{q\to 0\\ q\notin \bar{D}}} H(r^q) = F^S(r^0, G, U) + \sigma P^S(r^0, G, V) - \frac{1}{2}U(r^0),$$

where r^0 is the radius-vector of the point 0 situated on the surface S. The same relationships hold for the function $E(r^q)$.

From the latter formulas it follows that:

$$H^+(r^0) - H^-(r^0) = U(r^0),$$
$$E^+(r^0) - E^-(r^0) = V(r^0). \tag{12}$$

Consequently, when passing across the surface S analogues for the integral of the Cauchy type undergo a jump, the value of the jump being proportional to the corresponding densities.

The aforementioned properties of analogues for the integral of the Cauchy type allow us to solve the problems of separating a variable electromagnetic field into different parts and of continuing it into arbitrary regions of a conducting medium, i.e. the fundamental problems in the theory of interpretation of geomagnetic fields.

3. Separation of Variable Electromagnetic Field into the External and Internal Parts (The Generalized Kertz Method)

One of the first problems resulting from an analysis of the natural variable electromagnetic field of the Earth, is its separation into external and internal parts. The separation of the field into external and internal parts, first of all, allows us to determine whether the heterogeneity observed in the distribution of the variable geomagnetic field at the surface of the Earth (the geomagnetic anomaly) is related to the inhomogeneity of ionospheric currents exciting an external field, or to the heterogeneity in the structure of the Earth's interior. Thus the separated internal part of the geomagnetic field is the principal object for further investigations.

The fundamentals of the procedure of separating the geomagnetic field into external and internal parts was elaborated in the classical works of Gauss in application to the analysis of a field on a sphere.

These investigations have been developed lately in the following works (Vestine, 1941; Kertz, 1954; and Nedyalkov, 1965), where methods have been presented for the separation of the potential fields prescribed on an arbitrary surface of observation S. The technique elaborated above for the analogues for the integral of the Cauchy type permits us to extend these methods to variable electromagnetic fields.

Let E and H be electric and magnetic fields excited in a homogeneous unbounded space with conductivity σ by two systems of sources of arbitrary nature located within the regions D_1 and D_2 (Figure 1). The fields E and H can be represented in the form of the sum of two fields:

$$E = E^{(1)} + E^{(2)}, H = H^{(1)} + H^{(2)}, \tag{13}$$

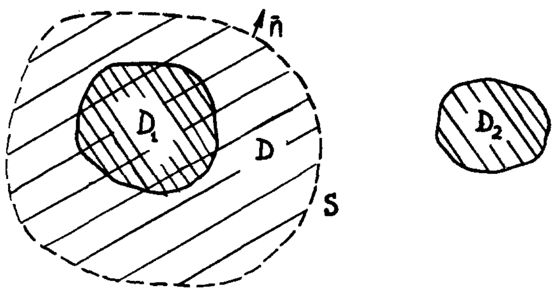

Fig. 1. The generalized Kertz method for separating a field into the external and internal parts.

where the components $E^{(1)}, H^{(1)}$ are related to the sources in D_1, and $E^{(2)}, H^{(2)}$ with the sources in D_2. The electromagnetic field at infinity satisfies the radiation condition.

We formulate the problem of the determination of the fields $E^{(1)}, H^{(1)}, E^{(2)}$ and $H^{(2)}$ from the given values of H and E at the surface S, i.e. we formulate the problem for the separation of the electromagnetic field into parts related to the sources located on different sides of the surface S (the positive direction of the normal to S is from the region D_1 to D_2; the region confined by S we denote as D).

We calculate the integral of the Cauchy type:

$$H(r^q) = F^S(r^q, G, H) + \sigma P^S(r^q, G, E). \tag{14}$$

In conformity to (9):

$$H(r^q) = \begin{cases} H^{(2)}(r^q)\ ; q \in D \\ -H^{(1)}(r^q)\ ; q \notin \bar{D} \end{cases}; \tag{15}$$

Since the fields $H^{(1)}$ and $H^{(2)}$ are continuous in the neighbourhood of S, then

$$\begin{aligned} H^+(r^0) &= H^{(2)}(r^0), \\ H^-(r^0) &= -H^{(1)}(r^0). \end{aligned} \tag{16}$$

Substituting formulas (11) into (16) we obtain:

$$\begin{aligned} H^{(1)}(r^0) &= \frac{1}{2}H(r^0) - F^S(r^0, G, H) - \sigma P^S(r^0, G, E), \\ H^{(2)}(r^0) &= \frac{1}{2}H(r^0) + F^S(r^0, G, H) + \sigma P^S(r^0, G, E), \end{aligned} \tag{17}$$

and in a similar manner for the electric fields:

$$E^{(1)}(r^0) = \frac{1}{2}E(r^0) - F^S(r^0, G, E) - i\omega\mu P^S(r^0, G, H),$$
$$E^{(2)}(r^0) = \frac{1}{2}E(r^0) + F^S(r^0, G, E) + i\omega\mu P^S(r^0, G, H). \tag{18}$$

It is (17) and (18) that give the solution to the problem pointed out in the title of this section.

The method developed (which may be called the generalized Kertz method) can be used widely in solving inverse problems in the geoelectric sciences; for separating the total electromagnetic field of the Earth into the external and internal parts or into the contributions from various conducting zones within the Earth.

4. Separation of a Field into its Normal and Anomalous Parts

This problem is one of the central problems for the analysis of a field. In the electrical reconaissance methods using artificial fields the abovementioned problem is solved relatively easily since the prescribed source normal field can always be calculated, and the anomalous field is obtained by subtracting the latter from the observed field. In studying variations in the natural electromagnetic field of the Earth such an approach cannot be used, since our concepts of the sources of the field are very schematic. At the same time, for separating a variable geomagnetic field into normal and anomalous parts a method can be used representing a development of the Kertz method and based on the difference in the space distribution of the normal and anomalous fields caused by the fact that the sources of the normal and anomalous components of the field are disposed on different sides of the surface of observation.

To illustrate this method we consider a model consisting of two half-spaces Π^- (the upper) and Π^+ (the lower) divided by a piecewise smooth surface S (Figure 2). The half-spaces Π^- and Π^+ are characterized by conductivities

$$\sigma^{(l)}, l = 1, 2$$

where the index $l = 1$ refers to Π^-, and $l = 2$ to Π^+, and each has constant magnetic permeability μ_0.

In the lower half-space there is a region of inhomogeneity Q with some conductivity

$$\sigma(r^q) = \sigma^{(2)} + \Delta\sigma(r^q), \; q \in Q, \tag{19}$$

different from that in Π^+, where q is the point of observation. The electromagnetic field is excited in the medium by an arbitrary system of sources located in the region P, in the upper half-space.

Considering a quasi-steady monochromatic field we shall write the main equations for

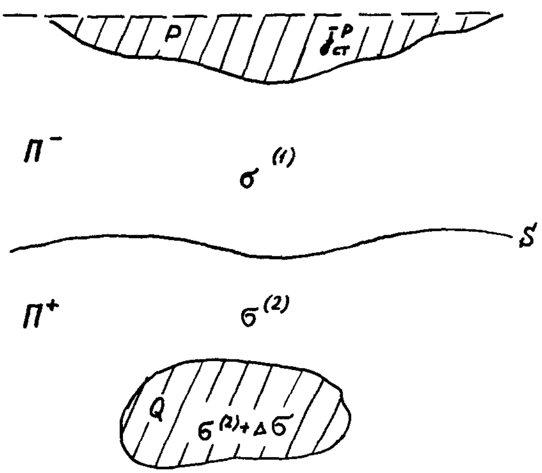

Fig. 2. The model of the medium consisting of two homogeneous half-spaces Π^+ and Π^- separated by the surface S. There is a region of inhomogeneity Q in Π^+.

the vectors of the electric E and magnetic H fields.

At any point of the space, except the points belonging to the regions P and Q the Maxwell equations are valid:

$$\begin{cases} \operatorname{curl} \boldsymbol{H} = \sigma^{(l)} \boldsymbol{E} \\ \operatorname{curl} \boldsymbol{E} = i\omega\mu_0 \boldsymbol{H} \end{cases} \tag{20}$$

where

$$l = 1 \text{ at } q \in \Pi^- - P, \text{ and } l = 2 \text{ at } q \in \Pi^+ - Q.$$

In the region P:

$$\begin{aligned} \operatorname{curl} \boldsymbol{H} &= \sigma^{(1)} \boldsymbol{E} + \boldsymbol{j}^P, \\ \operatorname{curl} \boldsymbol{E} &= i\omega\mu_0 \boldsymbol{H}, \end{aligned} \tag{21}$$

where $\boldsymbol{j}^P$ is the density of external currents.

In the region Q:

$$\begin{aligned} \operatorname{curl} \boldsymbol{H} &= \sigma^{(2)} \boldsymbol{E} + \Delta\sigma \boldsymbol{E} = \sigma^{(2)} \boldsymbol{E} + \boldsymbol{j}^Q, \\ \operatorname{curl} \boldsymbol{E} &= i\omega\mu_0 \boldsymbol{H}, \end{aligned} \tag{22}$$

where $\boldsymbol{j}^Q$ is the density of excessive electric currents in the region Q.

Thus, relationship (22) makes it possible to consider the lower half-space Π^+ as everywhere uniform, the presence of the region Q being taken into account by means of the excessive currents and charges.

The electric and magnetic fields can be represented in the form of the sum of two fields, i.e. the normal and anomalous fields:

$$E = E_n + E_a\ , \quad H = H_n + H_a \tag{23}$$

The normal part describes the field excited by external currents in the absence of an inhomogeneity ($\Delta\sigma = 0$) and the anomalous part – the field caused by the inhomogeneity. In other words, the anomalous field can be regarded as that of the charges and currents j^Q distributed over the region Q. This field propagates in a two-layered medium.

The normal field is represented in the form of two fields – the primary $(\underset{(1)}{H}_n, \underset{(1)}{E}_n)$ and secondary $(\underset{(2)}{H}_n, \underset{(2)}{E}_n)$:

$$H_n = \underset{(1)}{H}_n + \underset{(2)}{H}_n, \quad E_n = \underset{(1)}{E}_n + \underset{(2)}{E}_n\ , \tag{24}$$

where the primary normal field characterizes a field of external currents in the homogeneous space with the parameter $\sigma^{(1)}$, and the secondary field – a field of the currents induced in the half-space Π^+. In a similar way, the anomalous field can be represented in the form of the sum of the 'primary' $(\underset{(1)}{H}_a, \underset{(1)}{E}_a)$ and 'secondary' $(\underset{(2)}{H}_a, \underset{(2)}{E}_a)$ fields:

$$H_a = \underset{(1)}{H}_a + \underset{(2)}{H}_a, \quad E_a = \underset{(1)}{E}_a + \underset{(2)}{E}_a\ , \tag{25}$$

where the primary anomalous field characterizes the field of the charges and currents induced in the region Q and propagating in the homogeneous space with the parameter $\sigma^{(2)}$ and the secondary field is the anomalous field resulting from the interface S. Consequently, the 'primary' anomalous field is that excited by the inhomogeneity Q in the unbounded homogeneous space with the parameter $\sigma^{(2)}$, i.e. it is 'pure anomaly', which is not complicated either by the external sources, or by the influence of the interface S, therefore the separation of this field considerably simplifies the solution of the problem concerning the determination of the heterogeneity region Q. From Equations (23), (24) and (25) it follows that the field observed at the surface S can be represented in the form of the sum

$$H = \underset{(1)}{H}_n + \underset{(2)}{H}_n + \underset{(1)}{H}_a + \underset{(2)}{H}_a \tag{26}$$

and similary for the electric fields. Now we consider methods for the separation of the total electromagnetic field into the normal and anomalous parts and for the separation of the primary normal and anomalous fields under the condition that the parameters of the half-spaces Π^- and Π^+ $(\sigma^{(1)}, \sigma^{(2)})$ are known.

It is easy to obtain a solution to this problem using the general method of separating the fields outlined in Section 3. Specifically, by using formulas (17) for the fields prescribed at the upper (marked by the index '–') and lower (marked by the index '+') sides of the surface S, we obtain:

$$\begin{aligned} H_a^-(r^q) &= 2F^S(r^q, G^{(1)}, H_a^-) + 2\sigma^{(1)}P^S(r^q, G^{(1)}, E_a^-), \\ H_n^+(r^q) &= -2F^S(r^q, G^{(2)}, H_n^+) - 2\sigma^{(2)}P^S(r^q, G^{(2)}, E_n^+) \end{aligned} \tag{27}$$

where $q \in S$. (Indices (1) and (2) at the Green functions denote that the latter are taken either for the upper half-space with the wave number K_1, or for the lower one with the wave number K_2, respectively). Similar formulas for the electric field are obtained by interchanging H and E and replacing $\sigma^{(1,2)}$ by $i\omega\mu_0$.

At the interface the following conditions are fulfilled:

$$E^+ = E^- + \frac{\sigma^{(1)} - \sigma^{(2)}}{\sigma^{(2)}}(E^-, n)n, \qquad H^+ = H^-, \tag{28}$$

where n is the unit vector of the normal to the surface S directed to the lower half-space.

With allowance for (28), Equation (27) can be reduced to the form:

$$\begin{aligned} H(r^q) = & -2F^S(r^q, G^{(2)}, H - H_a) + 2F^S(r^q, G^{(1)}, H_a) - \\ & - 2\sigma^{(2)}P^S(r^q, G^{(2)}, E - E_a) + 2\sigma^{(1)}P^S(r^q, G^{(1)}, E_a). \end{aligned} \tag{29}$$

An expression of similar form is also obtained for the electric field.

These vector equalities form a system of 6 integral equations involving 6 unknown functions (E_{xa}, E_{ya}, E_{za}, H_{xa}, H_{ya}, H_{za}). So, the problem of the separation of the total field into normal and anomalous parts is, in the general case, reduced to the solution of a system of integral equations. Methods for solving this problem for plane and spherical surfaces S are described in the following works: Zhdanov (1973b), Berdichevsky and Zhdanov (1973, 1974).

At the same time, the problem of selecting the primary normal and anomalous fields is solved directly using integrals (17):

$$\begin{aligned} \underset{(1)}{H}_n(r^q) &= \frac{1}{2}H(r^q) - F^S(r^q, G^{(1)}, H) - \sigma^{(1)}P^S(r^q, G^{(1)}, E), \\ \underset{(1)}{H}_a(r^q) &= \frac{1}{2}H(r^q) + F^S(r^q, G^{(2)}, H) + \sigma^{(2)}P^S(r^q, G^{(2)}, E), \end{aligned} \tag{30}$$

and by analogy for the electrical fields.

For example, if S is the surface separating the non-conductive atmosphere ($\sigma^{(1)} = 0$) from the homogeneous conductive Earth ($\sigma^{(2)} = \sigma$) containing the heterogeneity region Q, then formulas (30) can be written as:

$$\begin{aligned} \underset{(1)}{H}_n(r^q) &= \frac{1}{2}H(r^q) + \frac{1}{4\pi}\iint_S \{(n, H)\mathrm{grad}_\mu G_0 + [[n \times H] \times \mathrm{grad}_\mu G_0]\}\, dS^\mu, \\ \underset{1}{H}_a(r^q) &= \frac{1}{2}H(r^q) - \frac{1}{4\pi}\iint_S \Big\{(n, H)\mathrm{grad}_\mu G + [[n \times H] \times \mathrm{grad}_\mu G]\Big\}\, dS^\mu - \\ &\quad - \frac{\sigma}{4\pi}\iint_S [n \times E] G\, dS^\mu; \end{aligned} \tag{31}$$

where $G_0 = 1/|r^q - r^\mu|$ is the Green's function for the Laplace equation and where $G = \exp(-K|r^q - r^\mu|)/|r^q - r^\mu|$ is the Green's function for the Helmholtz equation.

Thus, the primary normal magnetic field is determined directly from the magnetic field at S without any additional information, and to obtain the primary anomalous magnetic field it is necessary to know the conductivity σ of the homogeneous part of the Earth and, in the general case the electrical field at an arbitrarily shaped surface S.

5. Separation of Geomagnetic Anomalies into the Surface and Deep Parts

Electromagnetic anomalies can be subdivided in accordance with the nature of their heterogeneities into two groups: (1) the surface anomalies caused by electrical inhomogeneity of the near-surface layer of the Earth; (2) the deep anomalies related to the action of conductive zones in the Earth's crust and upper mantle*).

In interpreting electromagnetic anomalies one has, first of all, to determine to which of the above groups they belong. Most frequently, in practice both types of anomalies are observed simultaneously, i.e. the anomalous field represents the effect of two sources electromagnetically interrelated. The latter circumstance leads to essential difficulties in obtaining results from depth, since shallow inhomogeneities distort the field under observation. Therefore, the separation of electromagnetic anomalies into the surface and deep part is the challenge in the theory of interpretation. The principles of solving this problem are later set forth on the basis of the general theory for the separation of fields described above. The main idea of this method consists in the fact that the electromagnetic field being observed is related to three systems of extrinsic currents: (a) a system of Ionospherical currents, (b) a system of the currents induced in the near-surface inhomogeneous layer, (c) a system of the excessive currents filling the deep geoelectric inhomogeneities. The separation of these fields occurs by means of separate determination of the components stipulated by every one of these sources.

To illustrate the method we change the model of the Earth considered in Section 4 supposing that at the interface, i.e. at the surface S there is a thin conductive Price sheet with the surface conductivity $\xi(r^\mu)$, $\mu \in S$ continuously varying along S (Figure 3). The upper half-space Π^- is an insulator. As in Section 4, we suppose that the medium below the sheet S is homogeneous, but that in D the conductivity varies according to an arbitrary law:

$$\sigma^{(2)}(r^q) = \begin{cases} \sigma = \text{const}\,, & q \notin D\,, \\ \sigma + \Delta\sigma(r^q)\,, & q \in D\,. \end{cases}$$

* This classification differs from that of Schmucker (Schmucker, 1964) by the fact that transitional and deep anomalies are united in one class of deep anomalies. Such a unification is stipulated by the method for the interpretation of transitional and deep anomalies that can be elaborated using the same principles and differs from the method for the interpretation of surface anomalies.

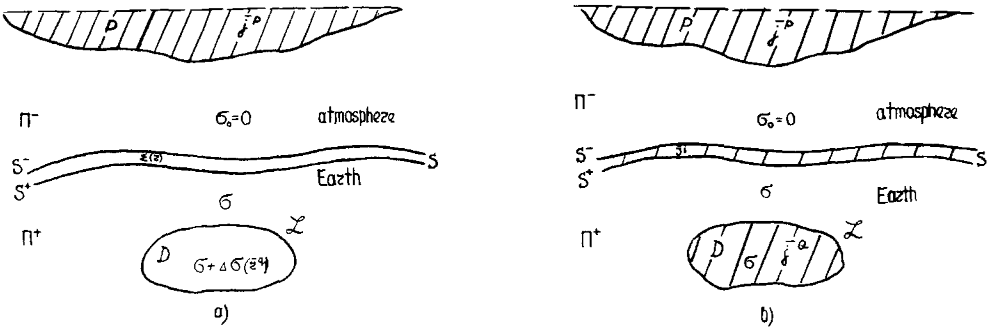

Fig. 3. (a) The model of the Earth containing surface and deep geoelectric inhomogeneities and excited by the currents j^P; (b) the model of the homogeneous Earth excited by the currents j^P, j^D, J^S,

In such a model the region D models a deep geoelectric inhomogeneity, and the sheet S approximates the near-surface inhomogeneous layer of the Earth. As above, a field in the model is excited by the external currents j^P in the region $P \subset \Pi^-$. The electromagnetic field in the model satisfies the equations:
(1) in Π^-

$$\text{curl}\, H = j^P \quad , \qquad \text{curl}\, E = i\omega\mu_0 H \, . \tag{32}$$

(2) in Π^+

$$\text{curl}\, H = \sigma^{(2)}(r^q)E \, , \qquad \text{curl}\, E = i\omega\mu_0 H \, . \tag{33}$$

At the sheet S the boundary conditions

$$\begin{aligned} &[n \times (H^+ - H^-)] = -\xi E_\tau \, , \\ &[n \times (E^+ - E^-)] = 0 \, . \end{aligned} \tag{34}$$

are fulfilled, where the indices '–' and '+' denote the field components on the upper and lower side of the surface S, E_τ is the electric field tangential component on S, n is the unit vector of the normal to S directed downwards. The tangential components of the electric and magnetic fields are continuous at the surface $\mathcal{L}$ confining the region D (Figure 3(a)). At infinity the fields satisfy the radiation condition.

Equations (33) and the boundary conditions (34) can be presented in the form

$$\text{curl}\, H = \sigma E + j^D , \qquad \text{curl}\, E = i\omega\mu_0 H ; \tag{35}$$

$$[n \times (H^+ - H^-)] = -J^S , \qquad [n \times (E^+ - E^-)] = 0 ; \tag{36}$$

where j^D and J^S are the volume and surface densities of the excessive currents in the region D and sheet S, respectively:

$$j^D = \Delta\sigma E\,, \qquad J^S = \xi E_\tau\,. \tag{37}$$

Therefore, the model of the inhomogeneous Earth excited by the currents j^P (Figure 3a) is equivalent to that of the homogeneous Earth excited by the currents j^P, j^D, J^S. All heterogeneities are substituted by these excessive currents (Figure 3(b)). Consequently the electromagnetic field in the model can be represented in the form of the sum of the P, D and S-components caused by the excessive currents in the regions P, D and sheet S, respectively:

$$H = H^P + H^D + H^S, \qquad E = E^P + E^D + E^S\,. \tag{38}$$

According to the terminology accepted above the P-component of field represents the normal field and the sum of the D and S components gives the anomalous field

$$\begin{aligned} H_n &= H^P, \qquad H_a = H^D + H^S, \\ E_n &= E^P, \qquad E_a = E^D + E^S, \end{aligned} \tag{39}$$

Thus the D-component can naturally be identified with deep anomalies, and the S-component with surface anomalies. The problem describing the separation of electromagnetic anomalies into the surface and deep ones is therefore reduced to the separate determination of the D and S components. Let us solve this problem.

If the total fields on the upper side of the surface S^-, viz.: H^-, E^- are known, then it will be possible to separate the total field into the normal and anomalous parts using the procedure set forth in Section 4, in so far as the sources of the corresponding components are on different sides of the surface of observation S^-. Further, under the condition that the sheet conductance ξ can be assigned, it is possible to select the D and S field components. By using the boundary conditions (36) it is possible to calculate the fields H^+ and E^+ on the lower side S^+ of the sheet. After mapping the field on the surface S^+, the P and S sources turn out to be over the surface of observation S^+, and the D-sources under it. Consequently, if we use the method of separating the fields described in Section 4 to the H^+ and E^+ fields, then we shall obtain the 'anomalous' field H^*, E^* consisting only of the D-component:

$$H_n^* = H^P + H^S, \qquad E_n^* = E^P + E^S, \qquad H_a^* = H^D, \qquad E_a^* = E^D. \tag{40}$$

The S-component of the field in accordance with (39) and (40) is determined by simple substraction:

$$H^S = H_n^* - H_n, \qquad E^S = E_n^* - E_n \tag{41}$$

For the purpose of interpretational convenience, it is useful to select the primary depth anomalies $H^D{}_{(1)}$, $E^D{}_{(1)}$, i.e. the fields excited by the currents j^D in a homogeneous unbounded medium with the parameters σ, μ_0. This problem is solved directly using quadrature formulas of the type (30), if the fields $\boldsymbol{H}^+$, $\boldsymbol{E}^+$ are known on the lower side of the sheet.

What is the physical sense and practical meaning of the operation of separating the primary deep anomalies $\boldsymbol{H}^D{}_{(1)}, \boldsymbol{E}^D{}_{(1)}$?

According to definition, the primary deep anomaly is the field excited by the currents $\boldsymbol{j}^D$ in a homogeneous isotropic medium. The density of currents $\boldsymbol{j}^D$ is defined not only by the parameters of the deep inhomogeneity, but also by the external currents $\boldsymbol{j}^P$ and the surface inhomogeneity S. However, the currents $\boldsymbol{j}^D$ are localized in the space exclusively within the region D, therefore, the field $\boldsymbol{H}^D{}_{(1)}$, $\boldsymbol{E}^D{}_{(1)}$ allows us in principle to determine the geometry of deep inhomogeneities and their location in space. Thus, the effect of $\boldsymbol{j}^P$ and S on the field is preserved, but their sources are as if transferred in the space and are concentrated within region D. Consequently, we can consider the primary deep field as an anomaly in its 'pure' form, complicated neither by external sources (in the sense mentioned above), nor by the surface heterogeneities S. The selection of such a field is of convenience from the viewpoint of searching for the deep inhomogeneities D.

6. Analytic Continuation of Variable Electromagnetic Fields in a Conductive Medium

In interpreting gravitational and static magnetic anomalies the methods of analytic continuation are widely used and consist of the reconstruction of the field distribution within a domain from its values known at the surface of observation. These methods are a powerful tool for solving a number of inverse problems in gravimetry and magnetometry.

The possibility of transferring the ideas and methods of analytical continuation of potential fields to variable electromagnetic fields was first considered in the works of the Indian geophysicist Roy (Roy, 1968, 1969). We give below the general solution to this problem based on the technique of analogues for the integral of the Cauchy type.

The problem of the continuation of electromagnetic fields is, in the general case, formulated in the following manner.

Let D be a region in the half-space confined by two surfaces S and Σ. The following situations are possible:

(a) The surfaces S and Σ are closed, Σ being entirely within S (Figure 4(a)).

(b) The surface S is of infinite extent, separating the whole space into the lower Π^+ and upper Π^- half-space, and Σ is closed, Σ being entirely situated in Π^+ not intersecting with S (Figure 4(b)).

(c) Both the surfaces S and Σ are of infinite extent, Σ lying entirely in Π^+ not intersecting with S (Figure 4(c)).

Let the magnetic field $\boldsymbol{H}$ and electric field $\boldsymbol{E}$ satisfy everywhere in D the Helmholtz equations:

$$\Delta H - K^2 H = 0, \tag{42}$$

$$\Delta E - K^2 E = 0, \tag{43}$$

where $K^2 = - i\omega\mu_0\sigma = \text{const.}$

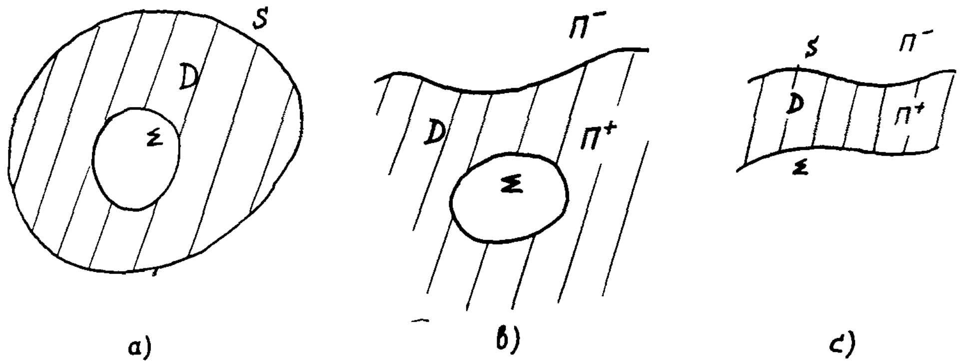

Fig. 4. On formulation of the problem on the continuation of electromagnetic field.

We assume that at the surface S the values of the H, E fields and their normal derivatives $\partial H/\partial n$, $\partial E/\partial n$, are known.

At infinity the radiation condition is fulfilled. It is necessary to determine the H, E, fields everywhere within D.

In such a formulation the problem of the continuation of electromagnetic field is reduced to the Cauchy boundary-value problem, the uniqueness of solution of which is well known. However, of importance in the continuation problem in the geoelectric science is the circumstance that the boundary Σ of the region D, within which the field satisfies Equations (42) and (43), is usually unknown. Moreover, the aim of continuation is frequently just the determination of this boundary.

Note, too, that the problem formulated above is related to the class of incorrectly posed problems of mathematical physics, in so far as infinitely large variations in solutions may result from small variations in the initial data. Therefore in the numerical continuation of electromagnetic fields the use of corresponding regularizing algorithms is needed.

Zhdanov (1973a) shows that the continuation problem can be reduced for a number of important cases to the spatial transformations of a field. Here we give the general scheme for elaborating such methods.

Note, first of all, that as a result of Maxwell's equations, the normal derivative of the magnetic field can be expressed in terms of the electric field and tangential derivatives of the magnetic field, and vice versa, the normal derivative of the electric field can be determined from the values of the magnetic field and the tangential derivatives of the electric field on S.

It is therefore possible to formulate the continuation problem just for the electromagnetic fields E, H given on S and continued in D, and this problem possesses a unique solution.

For obtaining the integral representations of the analytically continued electromagnetic field we shall use the technique of electromagnetic analogues for the integral of the Cauchy type. Consider the point q from D. In accordance with (9):

$$\begin{aligned} H(r^q) &= F^S(r^q, G, H) + \sigma P^S(r^q, G, E) + \\ &+ F^{\Sigma}(r^q, G, H) + \sigma P^{\Sigma}(r^q, G, E), \end{aligned} \tag{44}$$

where G is defined by formula (5) with the wavenumber K from (42) and (43).

Thus, for solving the problem of electromagnetic field continuation it is sufficient to calculate the integrals F^{Σ} and P^{Σ} in terms of field values at the surface S.

Suppose, that such a region $\gamma(q)$, $\Sigma \subset \gamma(q)$ exists, where the expansion:

$$G(r^q, r^\mu) = \int_\Omega f(r^q, \Omega)\, g(\Omega, r^\mu) \mathrm{d}\Omega \tag{45}$$

is valid, the functions $g(\Omega, r^\mu)$ everywhere in $\gamma(q)$ satisfying the Helmholtz equation:

$$\Delta_\mu g(\Omega, r^\mu) - K^2 g(\Omega, r^\mu) = 0. \tag{46}$$

Substituting (45) into the integrals F^{Σ}, P^{Σ} and changing the order of integration we obtain:

$$\begin{aligned} &F^{\Sigma}(r^q, G, H) + \sigma P^{\Sigma}(r^q, G, E) \\ &= \int_\Omega f(r^q, \Omega). \left\{ F^{\Sigma}(\Omega, g, H) + \sigma P^{\Sigma}(\Omega, g, E) \right\} \mathrm{d}\Omega. \end{aligned} \tag{47}$$

During this instant, according to (10) and under the radiation conditions to be fulfilled at infinity, by virtue of (46),

$$F^{\Sigma + S}(\Omega, g, H) + \sigma P^{\Sigma + S}(\Omega, g, E) = 0.$$

Hence:

$$F^{\Sigma}(\Omega, g, H) + \sigma P^{\Sigma}(\Omega, g, E) = -F^S(\Omega, g, H) - \sigma P^S(\Omega, g, E). \tag{48}$$

Consequently, expression (44) may be written in the form:

$$\begin{aligned} H(r^q) &= F^S(r^q, G, H) + \sigma P^S(r^q, G, E) - \\ &- \int_\Omega f(r^q, \Omega) \left\{ F^S(\Omega, g, H) + \sigma P^S(\Omega, g, E) \right\} \mathrm{d}\Omega. \end{aligned} \tag{49}$$

A similar formula for the electric field is obtained from (49) by substituting $\boldsymbol{E}$ for $\boldsymbol{H}$, $\boldsymbol{H}$ for $\boldsymbol{E}$ and σ for $i\omega\mu_0$. It is formula (49) that solves the problem for the analytic continuation of the field into the region D.

Therefore, the continuation problem is reduced to the search for the expressions of the type (45) that allow us to represent the Green's function in the form of a sum of products of two functions depending on the coordinates of the point of observation q and the coordinates of the point of integration μ. As is known, such expansions are in practice realized in those orthogonal coordinate systems, in which the variables in the Helmholtz equation are separable (Morse and Feshbach, 1953).

In the three-dimensional case there are 11 basic separable coordinate systems for the Helmholtz equation: (1) rectangular, (2) circular cylindrical, (3) elliptical cylindrical, (4) parabolic cylindrical, (5) spherical, (6) conical, (7) parabolic, (8) elongated spheroidal, (9) oblate spheroidal, (10) ellipsoidal, and (11) paraboloidal.

In the two-dimensional case for the Laplace equation ($K = 0$) all the coordinates obtained by conformal transformation of the rectangular coordinates are separable, and for the Helmholtz equation those coordinate systems are separable that are formed by confocal conical sections.

For $\gamma(q)$ it is therefore possible to take any region confined by the coordinate surfaces (or by the lines in the two-dimensional case) of that coordinate system, in which the variables in the Helmholtz equation are separable.

As an example, we consider a three-dimensional situation (c) with the surface S extended to infinity and with Σ – the horizontal plane (Figure 4(c)). In the rectangular Cartesian coordinate system expansion (45) for the Green's function takes the form (Morse and Feshbach, 1953),

$$G(r^q, r^\mu) = \frac{1}{|r^q - r^\mu|} \exp(-K|r^q - r^\mu|)$$
$$= \frac{1}{2\pi i} \iint_{-\infty}^{+\infty} \frac{1}{\nu} \exp\left\{i[\alpha(X^q - X^\mu) + \beta(Y^q - Y^\mu)] \mp \nu(Z^q - Z^\mu)\right\} d\alpha d\beta \tag{50}$$

where $\nu = \sqrt{\alpha^2 + \beta^2 + K^2}$, $\mathrm{Re}\,\nu > 0$.

The sign '–' in expansion (50) is taken under the condition $Z^q > Z^\mu$ and the sign '+ under the condition $Z^q < Z^\mu$. For our case $Z^q > Z^\mu$ (the Z axis is directed downwards) By comparing (45) to (50) we assume:

$$f(r^q, \Omega) = \frac{1}{2\pi i \nu} \exp[i(\alpha X^q + \beta Y^q) - \nu Z^q]$$
$$g(\Omega, r^\mu) = \exp[-i(\alpha X^\mu + \beta Y^\mu) + \nu Z^\mu]. \tag{51}$$

Substituting (51) into (49) we obtain:

$$H(r^q) = F^S(r^q, G, H) + \sigma P^S(r^q, G, E) + \\ + \frac{1}{8\pi^2} \iint_{-\infty}^{\infty} \frac{1}{\nu} \exp(-\nu Z^q) \exp[i(\alpha X^q + \beta Y^q)] \times \\ \times \iint_S \left\{ (n, H)\Omega + [\,[n \times H \times \Omega] + i\sigma[n \times E] \right\} \times \\ \times \exp[-i(\alpha X^\mu + \beta Y^\mu)] \exp(\nu Z^\mu) dS^\mu d\alpha d\beta, \tag{52}$$

where $\Omega = (\alpha, \beta, i\nu)$, n is directed into the lower half-space.

A similar formula for the electric field can be obtained by replacing E by H, H by E and σ by $i\omega\mu_0$. These are the formulas that solve the problem of the electromagnetic field continuation from an arbitrary surface S into the region D up to the horizontal plane Σ.

Formulas (49) and (52) give only the formal solution to the problem in the sense that for using these it is necessary to know the accurate and continuous values of the fields at the surface of observation. For a practical application of these formulas it is necessary to extend them by their regularized approximated representations. The simplest, but at the same time rather effective, method of regularization consists in limiting the range of interpretation of the 'frequencies' α and β by a finite cut-off frequency Ω_b. Expression (52) thus takes the form:

$$H(r^q) \approx H^{\text{reg}}(r^q) = F^S(r^q, G, H^{(\delta)}) + \sigma P^S(r^q, G, E^{(\delta)}) + \\ + \frac{1}{8\pi^2} \int_{-\Omega_b}^{\Omega_b} \int_{-\Omega_b}^{\Omega_b} \iint_S \left\{ (n, H^{(\delta)})\Omega + [\,[n \times H^{(\delta)}] \times \Omega] + i\sigma[n \times E^{(\delta)}] \right\} \times \\ \times \frac{1}{\nu} \exp[-\nu(Z^q - Z^\mu)] \exp\left\{ i[\alpha(X^q - X^\mu) + \beta(Y^q - Y^\mu)] \right\} dS^\mu d\alpha d\beta, \tag{53}$$

where $H^{(\delta)}$, $E^{(\delta)}$ are approximate values of the electromagnetic fields assigned at the surface S:

$$H^{(\delta)}(r^\mu) = H(r^\mu) + \delta H(r^\mu),$$

$$E^{(\delta)}(r^\mu) = E(r^\mu) + \delta E(r^\mu),$$

δH, δE are observational errors.

The cut-off Ω_b can serve as a regularization parameter. To determine the optimum value of Ω_b, it is necessary to know about a value of interference:

$$\|\delta H\| \leqslant \delta, \qquad \|\sigma \delta E/K\| \leqslant \delta, \tag{54}$$

where

$$\|\delta H\| = \sqrt{\iint\limits_S |\delta H(r^\mu)|^2 \, \mathrm{d}S^\mu} \tag{55}$$

The optimum frequency Ω_b can then be found on the basis of the Tikhonov–Ivanov optimum principle (Ivanov, 1966; Strakhov, 1969):

$$[\|H^{(\delta)}(r^\mu) - H^{\mathrm{reg}}(r^\mu)\|^2 - \delta^2]^2 = \min. \tag{56}$$

It is easy to work out formulas similar to (49)–(56) for analytical expressions for the continuation of electromagnetic fields into the regions confined by the coordinate surfaces of the other separable coordinate systems enumerated above.

The aim of analytic continuation is, first of all, the detection of singular points, lines and surfaces, which by analogy with the methods of gravimetry and magnetometry can be regarded as effective sources of the anomalous field. The distribution of these effective sources reflects the geometry of the bodies with an excessive electrical conductivity.

Zhdanov (1975) shows that the type and position of singular points, lines and surfaces of the analytically continued electromagnetic field are closely related to the form of the surface of deep inhomogeneities. It is known that under certain simple situations they may coincide with the elements of these surfaces. In particular, the ribs of conducting insertions or the edges of infinitely thin screens are the branch lines of the field.

As a typical example, results could be presented for the continuation of the variable magnetic field in the model of Dmitriev and Zakharov (1968). The model consists of the nonconductive atmosphere in contact at $Z = 0$ with the conducting Earth, where an ideal conducting infinitely thin vertical strip exists at a certain depth. The field in the model is excited by a plane E-polarized wave propagating from above downwards. We give diagrams of the anomalous magnetic field vertical component real part and the field isolines in the vertical plane that are analytically continued into the lower half-space (Figure 5) (the diagrams for the imaginary part have similar form). As is clear, the field isolines are focused at the top of the conducting strip. This result corresponds to the general theory set forth Zhdanov (1975) in conformity to which the infinitely thin ideal conducting screen edges represent the singularities for the analytically continued electromagnetic field.

Thus, the analysis of the spatial distribution of variable geomagnetic fields permits us to determine the location and character of sources for the field under study. Therefore, the procedure for the interpretation of analytically continued values for deep electromagnetic anomalies is similar to those methods that are of use already for analyzing geopotential fields. In interpreting the analytically continued electromagnetic field values it is at the same time possible to use a number of important and very useful features that are the property of electromagnetic fields only.

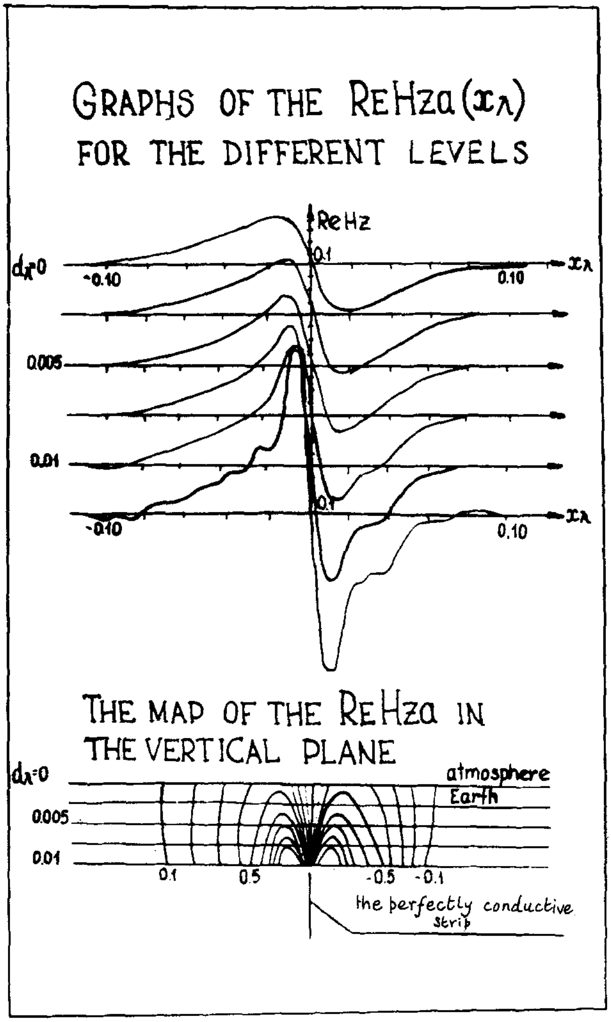

Fig. 5. The analytical continuation of the anomalies of variable magnetic field over an infinitely thin ideal conducting strip (the graphs and isolines of Re H_Z). 1 is the anomaly of variable magnetic field; 2 are the isolines of anomalous field in the vertical plane obtained as a result of analytical continuation.

As an example illustrating some of those features we shall consider the two following limiting cases.

(1) We form the streamlines for the electrical field in a homogeneous conducting medium containing a nonconductive insertion (as an example of such a model the section can consist of a conductive cover and a nonconductive base). Then by virtue of elliptical polarization of the field vectors (with dependence on time as: $\exp(-i\omega t)$) the streamlines at any point of the conducting medium will be in different directions at various moments of time, except for those coinciding with the nonconductive insertion contour (i.e. current at the conductor-insulator interface is parallel to the insulator surface). It is

expressed geometrically by the fact that streamlines for the real and imaginary parts of the *E* vectors describing the electrical field, are mutually intersecting and coincide only at the contour of a nonconductive body.

(2) A similar pattern occurs in the streamlines drawn for the magnetic field in a uniform conducting (or nonconductive) medium having insertions of infinitely large conductivity (the conducting zones in the Earth's crust and upper mantle may serve as an example of such a model). In this case the streamlines constructed for the real and imaginary parts of the *H* vectors describing the magnetic field are mutually intersecting everywhere in the conducting medium, except those streamlines which coincide with the conducting body contour (since the vector of magnetic induction directed along the normal to the surface of a conductor with infinitely large conductivity is equal to zero).

As an example, results will be given for the system of vector lines for a model, in which a well-conducting body buried in a conducting half-space is excited by a plane *E*-polarized wave that propagates from above downwards (Figure 6). As is clear from the figure, the isolines of the real and imaginary parts of the stream function ψ of the magnetic field *H* (i.e. the real and imaginary vector lines *H*) are reciprocally intersected everywhere in the conducting Earth and coincide only in the vicinity of the surface of the well-conducting body.

The examples considered here allow us to conclude that the interpretation of the analytically continued values of variable electromagnetic fields under the present approach can be made in the following two stages: (a) the construction of the streamlines for the real and imaginary parts of the vectors *H* and *E* (the 'real' and 'imaginary' streamlines), (b) the search for those curves (or surface in the three-dimensional case), where the 'real' and 'imaginary' streamlines coincide. It is thus desirable to continue the fields of various

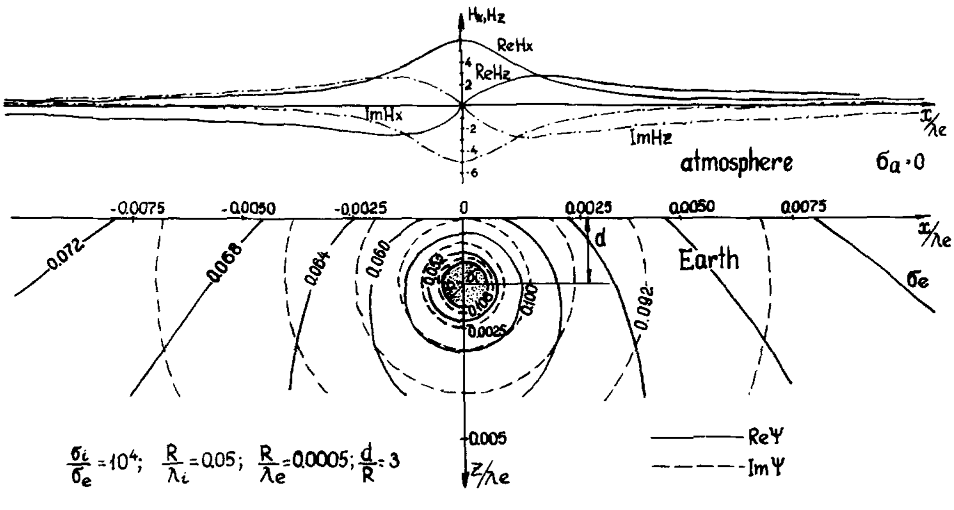

Fig. 6. The isolines of the real and imaginary parts of the stream function ψ and magnetic field *H* for the model, in which a well-conducting body immersed into the conducting half-space, is excited by a plane electromagnetic wave.

frequencies and to study the frequency dependence of the continued field streamline systems. This produces a more stable determination of the contours of conducting and insulating bodies, and minimises the observation and transformation errors. Note, finally, that by means of the continuation methods it is possible to recalculate downwards not only the values of the measured fields and stream functions, but also various parameters of the electromagnetic field (impedance, apparent resistance, etc.). Such calculations may be useful in the study of heterogeneous media.

As a result of the present review we may conclude that the technique of analogues for the integral of the Cauchy type allows us to extend a number of achievements in the theory of geopotential (gravitational and static magnetic) fields to time-variable electromagnetic fields. This opens a way for elaborating a unified approach to the problem of the interpretation of anomalies of gravitational, magnetic and variable electromagnetic fields of the Earth.

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