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MAGNETOTELLURIC MODELLING AND INVERSION IN THREE-DIMENSIONS

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Recent developments in three-dimensional modelling and inversion are discussed together with the progress which has been made in recent years. Emphasis is placed upon new work in inversion and large arrays, and on the type of exact solution of the direct problem which in our judgement has sufficient generality to make it potentially useful in three-dimensional inversion work. Initially, the equations of the problem are briefly established. Then, for the sake of argument, "modelling" and "automatic inversion" are defined as extreme aspects of the subject area. Typical case studies are included for lower dimensional configurations: new and robust methods of one-dimensional inversion are now available. The problem of large magnetometer arrays is mentioned. This is an area full of new and exciting possibilities for deep sounding and the exploration of anomalies. The theory, though, is only partly developed: available data have not yet revealed profiles as well resolved as those from a good combined electric and magnetic run at a single station, however, there have been some remarkable achievements. A new general attack upon the direct problem is reviewed, and we include some brief remarks on continental margins and how the edge effect problem is beginning to assume an optimistic shape through the method of matched asymptotic expansions. We conclude by mentioning other recent investigations in the field of modeling, the general trend of work over a wide field and some guesses about the future.

1. Introduction

In this series of workshops the inversion problem has been reviewed before: in 1972 for one dimension (Edinburgh) and in 1974 for two dimensions (Ottawa). Here, in 1976, we examine the problem for three dimensions, and look forward with interest to future reviews.

General two- and three- dimensional problems are intrinsically very difficult, but some excellent and elegant results have already been obtained. These are well worth reporting in detail. We shall link these new results with other strands of thought introduced for interest. But first, for those new to this area, we shall describe our terminology and notation.

2. Equations and terminology

An electrically conducting body is supposed to have a partly unknown electrical conductivity $\sigma(\mathbf{r})$, where $\mathbf{r}$ denotes the position vector of a field point. The body has many components: the earth, the ocean and (possibly) the ionosphere and magnetosphere. We are stationed at a point P on the earth's surface.

In one variation of the problem a known, i.e. "ordinary" electromagnetic field $\mathbf{E}_0$, $\mathbf{B}_0$, in the usual notation, is generated or, more likely, occurs naturally, at various frequencies ω. The induced fields $\mathbf{E}$, $\mathbf{B}$ at P are observed, where $\mathbf{E}_0 + \mathbf{E}$, $\mathbf{B}_0 + \mathbf{B}$ represents the total field at this frequency. We are required to make deductions about $\sigma(\mathbf{r})$ from these observations. There is some suspicion [42] that not much can be deduced about $\sigma(\mathbf{r})$ in more than one dimension without employing a grid of observation points $P_1, P_2, \ldots$ One also suspects that from a series of stations distributed along a straight line one will only gain two-dimensional insight into $\sigma(\mathbf{r})$. Probably a pretty large lattice of stations needs to be spread in order to obtain some knowledge of $\sigma(\mathbf{r})$ in three dimensions; but how large, how many lattice points and within what frequency range?

Ideally one should know what the source field is; more often only naturally occurring magnetic perturbations are available, and the source field is not known. In that case one can use surface integral techniques to separate the observed field into "external" and "internal" parts, or use ratios of the field to the field at some calibration point, or use ratios of components, or in addition make some simplifying assumptions, such as plane stratification, about the local geology. For example the original magnetotelluric method [45; 8] belongs to the last two classes. Let us assume for the sake of discussion that the electromagnetic field has been separated into its "external" or ordinary part $\mathbf{E}_0$, $\mathbf{B}_0$ and its "internal" or induced part $\mathbf{E}$, $\mathbf{B}$.

As the periods associated with the external field vary from seconds to days and years we may ignore displacement currents and use the pre-Maxwellian equations, which in rationalized m.k.s. units, with all variables proportional to $\exp(i\omega t)$, are

$$\mathbf{curl}\,\mathbf{E}_{\mathrm{TOT}} = -i\omega \mathbf{B}_{\mathrm{TOT}} \tag{1}$$

and

$$\mathbf{curl}\,\mathbf{B}_{\mathrm{TOT}} = \mu_0 \mathbf{J}. \tag{2}$$

Here, $\mathbf{E}_{\mathrm{TOT}}$ denotes the total electric field measured in fixed axes, $\mathbf{B}_{\mathrm{TOT}}$ the total magnetic field, $\mathbf{J}$ the electric current density and μ_0 the permeability of free space. Ohm's law $\mathbf{J} = \sigma \mathbf{E}_{\mathrm{TOT}}$, with a scalar conductivity $\sigma(\mathbf{r})$, is generally assumed to hold. Let $\mathbf{E}_0$, $\mathbf{B}_0$ be the external inducing electromagnetic field and $\mathbf{E}$, $\mathbf{B}$ the induced field so that

$$\mathbf{E}_{\mathrm{TOT}} = \mathbf{E} + \mathbf{E}_0 \tag{3}$$

and

$$\mathbf{B}_{\mathrm{TOT}} = \mathbf{B} + \mathbf{B}_0\,. \tag{4}$$

Combining Ohm's law with equations (1) to (4) and introducing the notation

$$k^2(\mathbf{r}) = i\omega\mu_0\sigma(\mathbf{r}) \tag{5}$$

gives within the conductor

$$\textbf{curl curl } \mathbf{E} + k^2\mathbf{E} = -k^2\mathbf{E}_0. \tag{6}$$

Outside the conductor

$$\textbf{curl curl } \mathbf{E} = 0\ . \tag{7}$$

These are the equations of the problem, presented in terms of the electric field. The interface conditions at the surface of the conductor are the usual ones: on occasion the ocean may be simulated by a surface conductor across which the tangential magnetic field components may be discontinuous. The boundary conditions, at infinity or otherwise, are chosen to suit the problem in hand.

In the *direct problem* $\sigma(\mathbf{r})$ is assumed to be known, and a solution for $\mathbf{E}$ has to be found.

In the *inverse problem* $\sigma(\mathbf{r})$ is to be deduced from the surface observations of $\mathbf{E}_{\mathrm{TOT}}$ and $\mathbf{B}_{\mathrm{TOT}}$. In all methods of inversion used to date it has been essential to be able to solve a direct problem of some kind: either analytically, symbolically or by computer.

3. The extremes: modelling and automatic inversion

At the ends of the spectrum of methods available to the geophysicist lie the two extremes of inversion, namely, *modelling* and *automatic inversion*. In this section these will be described as briefly as possible.

Modelling

Modelling is as much an art as a science. A perfect modern example of this is the recent investigation by Cox and Filloux [10] of the California electromagnetic coastal anomaly. A carefully specified conductivity model was assumed for a (two-dimensional) transverse electric mode problem: an ocean with a straight coastline was also included for good measure. This model was based on the authors' previous investigations of possible conductivity distributions. They solved the basic equations by computer, and compared their results with observations for various frequencies. The procedure yielded a reasonable spatial variation of the amplitudes and phases of various components. Results were presented for 0.25, 0.5, 1.0 and 2.0 cycles per hour. It is clear that the solution of the direct problem is of immediate interest when modelling.

Automatic inversion

At the other end of the scale lies automatic inversion. In principle one would like to produce a computer program which would read in the data and print out a set of consistent distributions of conductivity, together with information on the resolution attained. This procedure is more automatic than modelling, but not necessarily better.

The conductivity $\sigma(\mathbf{r})$ is partly unknown, but for a given $\sigma(\mathbf{r})$ equation (6) can in theory be solved for **E**. Then we may write

$$\mathbf{E} = \mathbf{E}\,[\sigma]\ ; \tag{8}$$

that is to say, **E** is a *functional* of $\sigma(\mathbf{r})$: also, **E** depends upon the frequency of excitation ω, but it is unnecessary to exhibit this dependence explicitly.

For the sake of argument we now consider any entity g which could be obtained from the electromagnetic field **E**, **B**. For example g could be an impedance like $|\mathbf{E}|/|\mathbf{B}|$, or a phase (say the phase of a particular component of **E**), or a ratio of two components (say, B_1/B_2 in some frame of reference): g could also be a three-vector whose components represented in turn the three quantities just mentioned. It could also be a multi-dimensional vector, or a tensor. One particularly good choice, involving much computation though, might be SCHMUCKER's matrix h_{mn} of transfer functions ([42] p.20) such that if suffixes 1, 2, 3 distinguish the three magnetic field components,

$$B_m = \sum_{n=1}^{3} h_{mn}\, B_{\text{on}}\,. \tag{9}$$

Then we would have $g = (h_{mn})$, a 3×3 matrix. We make only one restriction: g shall be an element in a space with norm $||.||$ (for a brief survey of this notation see RICHTMYER and MORTON ([36] p.31). It follows that for two elements g_1 and g_2 of the space,

$$||g_1 - g_2|| = 0 \text{ if and only if } g_1 = g_2. \tag{10}$$

Then equation (8) is included in the more general statement that

$$g = g[\sigma]\ ; \tag{11}$$

that is, g is a *functional* of $\sigma(\mathbf{r})$. Alternatively, we may write

$$g = g(\mathbf{r})\ ; \tag{12}$$

that is, g is a function of position.

An artificial example of such a functional is

$$g(\mathbf{r}) = g[\sigma] = \iiint \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx'\, dy'\, dz' . \tag{13}$$

This shows how one particular conductivity distribution $\sigma(\mathbf{r})$ gives a range of numbers $g(\mathbf{r})$, so that g itself is a function of position.

The data consists of a finite number of observations $g_1, g_2, \ldots, g_n$ of g, at various points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ on the surface of the conductor, for various frequencies. Unfortunately the authors are not in a position to advise on the features which distinguish good and bad data, but *the collection of good data seems to be an important feature of major investigations* [23].

There are likely to be infinitely many conductivity models $\sigma(\mathbf{r})$ such that

$$g(\mathbf{r}_1) = g_1,\; g(\mathbf{r}_2) = g_2, \;\ldots g(\mathbf{r}_n) = g_n \; ; \tag{14}$$

the problem is how to find them. One should also properly comment that equations (14) may be insoluble. In two dimensions there is, for any given smooth curve, an iterative method of finding that particular normal to it which passes through a given point not on the curve. BACKUS and GILBERT [4] generalized this to equations like (14), generating from some initial guess $\sigma_1(\mathbf{r})$ a sequence of approximations $\sigma_2, \sigma_3, \ldots$ which might converge to the function $\sigma(\mathbf{r})$. They also showed how to generate other solutions from this solution and in [5] discussed the uniqueness of the solution and the resolving power of the data. Their contribution [6] to the phenomenon of trade-off between accuracy and resolution offers an attractive format in which to discuss the real meaning of any approach to inversion. Their papers are well written, and easily converted from seismology to geomagnetism.

PARKER [29] applied the BACKUS—GILBERT format to reworking an extremely valuable pioneering determination by BANKS [7, 62] of a distribution of σ for a spherically symmetric earth. PARKER's helpful investigation gave deep insight, and set a bench mark for judging further investigations. PARKER's first iterative approach did not in fact converge; slight inconsistencies in the data may throw the Backus—Gilbert method off the scene, so that it may diverge. JUPP and VOZOFF [20] have shown how the method may be stabilized so that it will converge even for ill-posed problems. In that paper, and an accompanying one, under reversed authorship [50], they carry out thought experiments for a layered earth, and apply their method also to measurements above a sedimentary basin. They also use d.c. resistivity measurements in conjunction with the magnetotelluric method.

It hardly seems fair to include such a thorough investigation as LARSEN's [23] under the heading of "automatic inversion". But he has produced such a clear account, and good conductivity resolution that it must be described here. LARSEN used electric and magnetic field measurements taken on Hawaii, at what is virtually a single site. Twenty-two months of data converted to hourly values were used. The observations were of high quality, enabling an accurate inversion to be carried out using a hitherto unpublished method due to SCHMUCKER. The method has weighting factors built into it, so that unstable

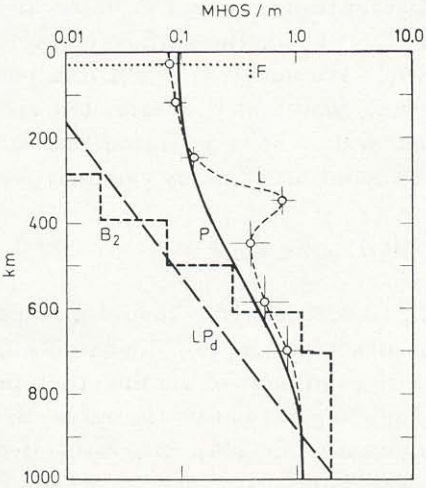

Fig. 1. (After LARSEN [23].) Locally smoothed conductivity profile L with spread (vertical bars) and conductivity errors (horizontal bars) compared with the FILLOUX profile F [11], the LAHIRI and PRICE [21] model d profile LP_d, the BANKS [62] profile B_2, and the PARKER [29] profile P

features can be quenched if they occur (e.g. the traditional one mentioned by WEIDELT [57, p.282], of a poor conductor sandwiched between two good conductors). The BACKUS—GILBERT technique was used to discuss the accuracy and resolution of the conductivity profile, which showed a well resolved highly conducting zone at a depth around 350 km. Apart from this layer the profile is similar to the reliable portion of Parker's [29] conductivity profile, as illustrated in Fig. 1.

The *uniqueness* of the solution for σ, originally proved by TIKHONOV [46], BAILEY [61] and — effectively — WEIDELT [57], has now been proved again by LOEWENTHAL [27].

WEIDELT [58] has recently made a novel, and strikingly simple approach to inversion for *two-dimensional* structures, again by linearizing the problem in the neighbourhood of a trial solution. He takes $\sigma(\mathbf{r})$ to be constant within m small rectangles. Since there are m linearized versions of equations (14) he obtains m equations in n-unknowns at each step of the solution. He solves these equations by the method of generalized matrix inversion, which itself leads naturally to problems of accuracy and resolution. It appears that Weidelt's method is capable of generalization in many ways and is *im Geist dei Geophysik.*

The whole problem of aligning models with observations lies in the field of design of experiments in non-linear situations where, often,

$$g[\sigma_1(\mathbf{r}) + \sigma_2(\mathbf{r})] \neq g[\sigma_1(\mathbf{r})] + g\,[\sigma_2(\mathbf{r})]\ . \tag{15}$$

The geophysicist is likely to find applications to fertilizer and feed consumption in the statistical journals, but a little effort of the imagination transforms agriculture into geophysics. Our system has a distributed parameter $\sigma(\mathbf{r})$, but once the method of discretizing $\sigma(\mathbf{r})$ has been decided, we have a lumped parameter system. A suitable book to study might be Applied Regression Analysis by DRAPER—SMITH [12].

Since we have mentioned the non-linear inversion problem in this section, we conclude with the comment that linear problems in geophysics are even more susceptible to *functional analytic* methods. These are useful for proving the existence of solutions, and can be used constructively to find solutions. HUTSON et al. [17] showed how to treat direct problems of induction by this means. DUCRUIX et al. [13] showed how to use these methods to continue potential fields from uneven surfaces onto flat reference surfaces. References to earlier work and to other authors may be found also in LE MOUEL et al. [25] where potential problems have been solved by functional analytic methods.

4. Magnetometer arrays

When confronted by actual data from magnetometer arrays, geophysicists are forced towards very difficult problems of three-dimensional inversion. The simplest assumption to make is that the array lies above a horizontally stratified earth, but that the source field is such as to cause telluric curents to flow in a complicated manner. Even this assumption does not make things easy.

The validity of the one-dimensional approximation used in early magnetotellurics was questioned by WAIT [51] and later by PRICE [34]. However, systematic use of a number of stations on a local rather than a global scale did not come until 1964 (SCHMUCKER [41]). A valuable review of work done in the embryonic decade 1950—1960 has been given by WAIT [53]. Following PRICE's pioneering work in 1950 [33] for a uniform half-space, WAIT [54] covered much ground for multilayered spaces; and after work by WHITHAM [60] and SRIVASTAVA [44] a major consolidation of the theory was achieved by SCHMUCKER [42]. His work was written with practical applications in mind, and has had a wide influence. LILLEY [26] neatly summarized the applications of the theory, drawing heavily on the notations of PRICE and SCHMUCKER. We refer the reader to his paper and that of LARSEN [23] for details.

Briefly, the separable solutions of the induction equation (6) with conductivity σ a function of z (depth below ground level) only are

$$\mathbf{E} = (P_x, -P_y, 0)Q \exp(i\omega t), \tag{16}$$

with the corresponding magnetic induction vector

$$\mathbf{B} = i\omega^{-1}(P_x Q_z, P_y Q_z, \nu^2 PQ) \exp(i\omega t). \tag{17}$$

Here, Q is a function of z only, P of x and y only, and the subscripts denote differentiation so that $P_x = \partial P/\partial x$ *etc*. The functions P and Q have to satisfy the equations

$$P_{xx} + P_{yy} + \nu^2 P = 0 \tag{18}$$

and

$$Q_{zz} - \{\nu^2 + k^2(z)\}Q = 0, \tag{19}$$

where ν is a constant arising from the separation of the variables.

Suppose that $k(z)$ is a constant within each one of n layers being equal to k_m in the mth layer (of thickness h_m), and we distinguish x, y and z components by suffixes 1, 2 and 3. Then we *define* θ_m by

$$\theta_m^2 = \nu^2 + k_m^2 \tag{20}$$

and $G_1(0)$ by

$$\frac{1}{G_1(0)} = \coth\left\{\theta_1 h_1 + \coth^{-1}\left[\frac{\theta_1}{\theta_2}\coth\left\{\theta_2 h_2 + \coth^{-1}\left[\frac{\theta_2}{\theta_3}\coth\left\{\theta_3 h_3 + \right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left. \ldots + \coth^{-1}\left[\frac{\theta_{n-1}}{\theta_n}\right]\right\}\right]\right\}\right]\right\}. \tag{21}$$

The bottom layer is of infinite thickness ($\coth \infty = 1$), and so only its conductivity enters the expression. In terms of these expressions the following ratios hold good:

$$E_1/B_2 = i\omega/\{\theta_1 G_1(0)\} \tag{22}$$

and

$$(B_{1x} + B_{2y})/B_3 = \theta_1 G_1(0). \tag{23}$$

In addition there are more complicated relationships involving P and its derivatives. We see that if the constant ν did not appear in equation (20), relationships (22) and (23) would hold for all electromagnetic fields. The presence of ν is unwelcome. LILLEY points out that P cannot be simply resolved into spatial Fourier harmonics (which would then give from equation (18) a ν for each harmonic) because it is difficult to distinguish between standing waves and running waves. He shows how magnetometer arrays in which "tens of instruments simultaneously record magnetic fluctuations over areas of order 10^5 km^2" may be used to sort out the type of wave. He considers also that "horizontal field gradient values . . . for horizontal layering . . . may be some of the most useful data produced by an array study". He would use these in formulae such as (23), and has already carried out a separation of wave modes on a south-east Australian array. In a forthcoming note in the Journal of Geomagnetism and Geoelectricity he has succeeded with M. N. SLOANE in carrying out an inversion to obtain a preliminary estimate of the conductivity.

Of course, we have already seen in LARSEN's work [23] how one station can be sufficient to extract the conductivity distribution for a layered structure. The exciting part about large arrays is the possibility of entering them for the three-dimensional inversion races. LILLY's work is part of the necessary scientific spadework, and the intention is no doubt to vary the assumption of horizontal stratification.

Meanwhile, in the north-western United States and south-western Canada, CAMFIELD and GOUGH [9] have been carefully studying the results from a widely spaced array of thirty-one magnetometers. They have been particularly concerned with anomalies, and have given these a thorough physical discussion by studying the maps of Fourier transform amplitudes at various frequencies. They have also carried out modelling of $\sigma(z)$ and come up with the complicated structure shown in Fig. 2. This diagram illustrates the considerable progress made possible by the judicious use of models.

Another feature of CAMFIELD and GOUGH's array studies is the appearance on their maps of anomalies which appear to be due to physical features off the maps. They suggest for these the name *Vartran* anomalies, for "variable transmission". They have not attempted a major physical interpretation of these, and it would be interesting to see a discussion in terms of magnetic field line topology, extending methods like those in LAWRIE [24], HYNDMAN and COCHRANE [18] or HEWSON-BROWNE and KENDALL [15]. It is certainly true that regions of high conductivity would tend to exclude magnetic fields travelling at certain speeds, and that every non-uniformity in σ has associated with it a speed at which the field lines will suffer maximum drag.

BABOUR et al. [3] report another large array of 60 stations operating in the Northern Pyrenees from 1972 to 1974. The periods were measured in the range from a few minutes to a few hours. As their report has not yet been published, it is difficult to relate their inferences that the induced electric currents appeared to have a distribution independent of time to the work of LILLEY and of SCHMUCKER (*loc. cit.*). We look forward to the appearance of their work.

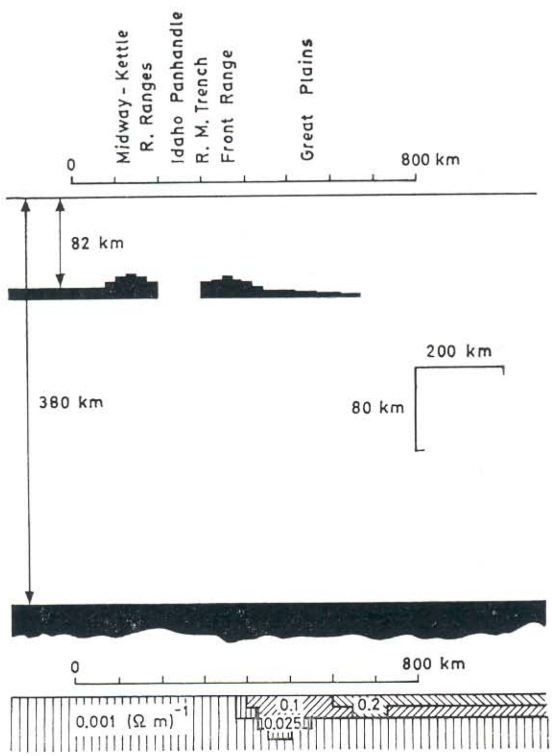

Fig. 2. (After CAMFIELD—GOUGH [9].) The upper section represents the upper mantle conductive layer, with 2.5-fold vertical exaggeration, as an equivalent thickness at $\sigma = 0.2$ (ohm m)$^{-1}$; all black parts of this figure have this conductivity, which extends to infinite depth. The conductivity model for the sedimentary layer is shown at lower right, with 25-fold vertical exaggeration

5. Three-dimensional modelling

Nature invents such a variety of configurations that it would be possible to spend a good many years legitimately studying the electromagnetic responses of discs, ellipsoids, wedges and other shapes. Fortunately this topic has been long studied [22] and reviewed recently [16] see also [38, 52], so we can concentrate on problems with some generality in them. It is important though, for workers in this area to keep on producing exact solutions wherever possible, as they will always be useful.

Accordingly, we concentrate on (i) a recent method of computing the effects of anomalies, and (ii) a method of solving problems involving the edge of a finitely conducting ocean.

Three-dimensional anomalies

WEIDELT [58] suggests that the next class of problems capable of solution by computer in a finite time is as follows. Within a horizontally stratified half-space S lies a single domain R (which he calls *anomalous)* whose conductivity is non-uniform and different from that of S. The problem is to calculate the extra response of R, and RAICHE [35] has shown how to do this by using Green's tensors. TREUMANN [47, 48] has used Green's functions for two dimensions. (One can speculate that the Green's tensor solution might be useful for *automatic inversion*). Suppose that equation (6) (omitting equation (7) for brevity) has been solved for a *normal* conductivity distribution $\sigma_n(\mathbf{r})$ in the absence of the anomalous region R. Let the solution be $\mathbf{E}_n$ so that if $k_n^2 = i\omega\mu_0\sigma_n$

$$\textbf{curl curl } \mathbf{E}_n + k_n^2\mathbf{E}_n = -k_n^2\mathbf{E}_0. \tag{24}$$

Then defining the anomalous induced electric field by $\mathbf{E}_a = \mathbf{E} - \mathbf{E}_n$ and conductivity by $\sigma_a = \sigma - \sigma_n$, equations (6) and (24) give

$$\textbf{curl curl } \mathbf{E}_a + k_n^2\mathbf{E}_a = -k_a^2(\mathbf{E}_0 + \mathbf{E}_n + \mathbf{E}_a) , \tag{25}$$

where $k_a^2 = i\omega\mu_0\sigma_a$. In these equations σ_n and σ_a are known scalars and σ_a will be zero everywhere, except in the anomalous region R. In R the conductivity will be $\sigma_n + \sigma_a$. The vector field $\mathbf{E}_0$ is the ordinary field which induces telluric currents to flow, and $\mathbf{E}_n$ can be found from it *via* equation (24). So the only unknown in equation (25) is the vector field $\mathbf{E}_a$ which appears on both the left and right hand sides. Introducing a Green's tensor $G(\mathbf{r}|\mathbf{r}')$ in the usual notation, equation (25) can be written as an integral equation

$$\mathbf{E}_a = -{}_R\!\int k_a^2(\mathbf{r}')G(\mathbf{r}|\mathbf{r}') \cdot (\mathbf{E}_0 + \mathbf{E}_n + \mathbf{E}_a)dx'\, dy'\, dz'. \tag{26}$$

Here, the Green's tensor is for the operator on the left-hand side of equation (25). There are certain advantages in using an equation such as (26), for $\sigma_a(\mathbf{r})$ vanishes outside the anomalous region R, and the integration is therefore confined to this finite region. Moreover, the Green's tensor within the normal layered structure can be constructed analytically; here WEIDELT quotes SOMMERFELD [43] and WAIT [55]. He also has to do considerable extra work! Equation (26) is then soluble by numerical methods. One should notice how the region over which the computation extends has been reduced from an infinite one (cf. [19]) to the finite one R at the expense of the simplicity of the finite-difference technique but with a gain of integrity. Both treatments are necessary and important complementary methods.

WEIDELT has validated this technique upon various buried anomalies. Satisfactory convergence of a numerical Gauss—Seidel method of solution of

(26) occurs at conductivity contrasts of up to 1 : 100. The method offers the chance of carrying out thought experiments such as that shown in Fig. 3 in which a transition is effected from a two-dimensional problem to one of three dimensions. It is a promising competitor for finite-difference techniques.

Continental margins

Very many investigations have to cope with the edge effect due to sedimentary layers or the presence of the ocean. This effect often extends a long way inland [18, 63, 9, 10, 39], so it is worth a brief discussion. Following the discovery of the Parkinson vector [30, 31, 32] RIKITAKE [37] formulated an early model of the Pacific ocean as a hemispherical shell of infinite conductivity. ASHOUR [1, 2] then obtained an elegant exact solution to this problem, and confirmed theoretically the gross effect predicted by PARKINSON. The actual ocean, though, is of finite conductivity, and the problem of a resistive shell over an infinitely conducting sphere has proved less tractable. By using physical arguments based on the merging and decay of magnetic fields HEWSON-BROWNE and KENDALL [15] and HEWSON-BROWNE [14] have shown how to evaluate the oceanic electric currents flowing near the edge of an ocean of *arbitrary* shape and arbitrary marginal edge-slope. We are optimistic that thin highly conducting regions may soon be regarded as normal, rather than anomalous.

Extending and simplifying our earlier analysis we first solve the edge problem for an *infinitely conducting* ocean which is large, but of arbitrary shape. We then extend the problem to the case of finite conductivity. Provided that the curvature of the coastline is large compared with the shielding depth $P/2$, the method of matched asymptotic expansions [49] can — in principle, at any rate — be used. We concentrate on the first order solution only.

First, solve the problem of the source currents or fields, passing above an infinitely conducting sphere coincident with the earth's mean surface (a sphere of radius R_E). This will give the following components of a horizontal magnetic induction vector: B_T^∞ parallel to the local mean coastline, and B_N^∞ at right angles to the local mean coastline.

Second, solve the problem of the source currents or fields, passing above an infinitely conducting sphere of radius $R_E - P/2$ in the absence of the ocean. Let the value of the *downwards* vertical magnetic induction vector at the point where the edge of the coast would lie be B_{EXT}^∞, a constant.

The value B_T^∞ should be near its true value and perturbations to it are probably best obtained after matching to the two-dimensional inner problem. Then using a local system (u, v) of rectangular Cartesian co-ordinates with the u-axis horizontal and parallel to the coastline and the v-axis vertical, for the

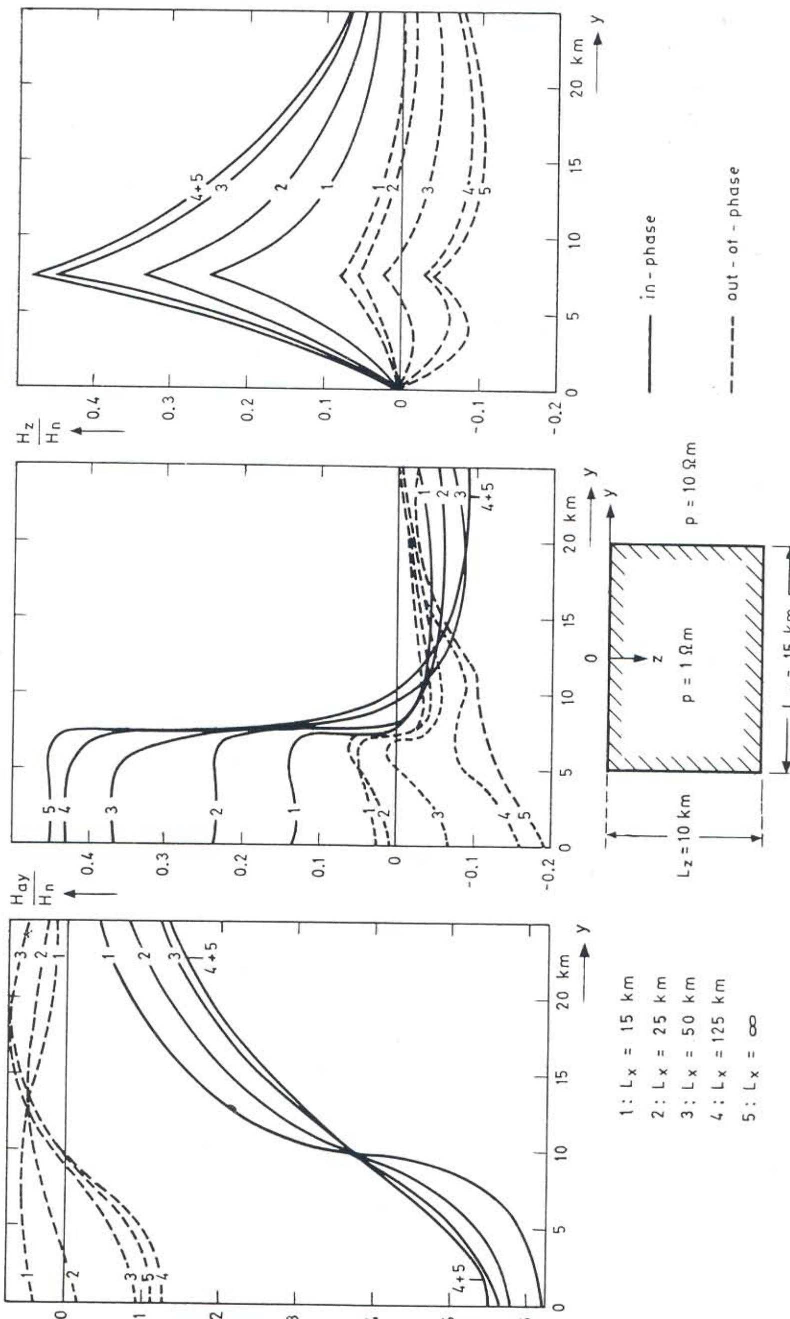

Fig. 3. (After WIEDELT [59].) The field components on a central profile across a rectangular domain with dimensions L_x, L_y, L_z are shown, and the transition process from three to two dimensions as L_x increases to infinity

solution of the inner problem, we choose the complex potential in the complex w-plane, where $w = u + iv$ to be, in the *absence* of the ocean,

$$\Phi^\infty = P^{-1} B^\infty_{\mathrm{EXT}} \left(w + \frac{1}{2} i P\right)^2 + B^\infty_N w , \tag{27}$$

where B_1 and B_2, the u and v components of **B** are given by

$$B_1 - iB_2 = d\Phi^\infty/dw . \tag{28}$$

The solution is such that B_1 vanishes at $u = -(1/2)\, PB^\infty_N / B^\infty_{\mathrm{EXT}}$. A sketch of the magnetic field lines in the *presence* of the infinitely conducting ocean is shown in Fig. 4.

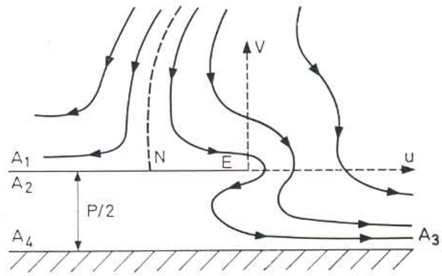

Fig. 4. (c.f. SCHMUCKER [42].) The underlying infinitely conducting model: A_1EA_2 is an infinitely conducting sheet, and the region below A_3A_4 is also of infinite conductivity. The magnetic field lines of the inner expansion are shown. The neutral point N is shown in the diagram for convenience only. It will usually be at a great distance from the end E of the sheet, and serves usually to provide a slight gradient in an incident field which would otherwise be uniform

Solving the complex variable problem as illustrated in Fig. 4 gives the inner solution near A_1 as

$$B_1 \sim -2B^\infty_{\mathrm{EXT}} P^{-1} \left(|u| + \frac{P}{2\pi} \log \frac{2\pi |u|}{P}\right) + B^\infty_N , \tag{29}$$

and near A_3 as

$$B_1 \sim 2B^\infty_{\mathrm{EXT}} P^{-1} \left(u - \frac{P}{2\pi} \log \frac{2\pi u}{P}\right) + B^\infty_N . \tag{30}$$

Following RODEN [40] and PARKER [28] we may develop an integro-differential equation for the electric field $E(u)$ within the strip A_1EA_2 when its surface conductivity $\varkappa(u)$ becomes finite. If B_2 is the component of magnetic induction in the v-direction, the induction equation gives

$$\partial E/\partial u = i\omega B_2 . \tag{31}$$

But B_2 can be found by using the formula "$\mu_0 I/(2\pi r)$", putting in equal and opposite image currents, and adding on what the value of B_2 would be if the strip A_1EA_2 were absent. Thus,

$$\frac{dE}{du} - \frac{i\omega\,\mu_0\,P^2}{2\pi}\int_{-\infty}^{0}\frac{\varkappa(u')\,E(u')\,du'}{(u-u')\{P_2+(u-u')^2\}} = -\,i\omega\,B_{\mathrm{EXT}}^{\infty}\,. \tag{32}$$

Integrating with respect to u leaves us with a free-floating constant — a constant of inspiration. This can be evaluated if $\varkappa \to 0$ as $u \to -\infty$. The asymptotic form of E is then

$$E_\infty = -\frac{1}{2}\,i\omega\,P\,(B_N^\infty + 2u\,B_{\mathrm{EXT}}^{\infty}\,P^{-1})\,, \tag{33}$$

and the integral equation is

$$E - \frac{i\omega\,\mu_0}{4\pi}\int_{-\infty}^{0}\varkappa(u')E(u')\log\left\{\frac{(u-u')}{(u-u')^2+P^2}\right\}du' =$$
$$= -\frac{1}{2}\,i\omega\,P\,(B_N^\infty + 2u\,B_{\mathrm{EXT}}^{\infty}\,P^{-1})\,. \tag{34}$$

On the other hand, it is arguable that the spirit of the present approximation is best fulfilled by ignoring $B_{\mathrm{EXT}}^{\infty}$. It then becomes possible to let $\varkappa \to \varkappa_\infty$ as $u \to -\infty$, where $\varkappa_\infty$ is a constant. The correct asymptotic form of E is then

$$E_\infty = -\frac{1}{2}\,i\omega P B_N^\infty \Big/ \left(1+\frac{1}{2}\,i\omega\mu_0\varkappa_\infty P\right)\,, \tag{35}$$

and the integral equation is simply (34) with $B_{\mathrm{EXT}}^{\infty} = 0$.

The problem considered by HEWSON-BROWNE [14] was the finite width strip problem, but our present, more general, formulation can be readily converted to his by taking

$$B_0 = B_{\mathrm{EXT}}^{\infty} \quad \text{and} \quad a = \frac{1}{2}\,PB_N^\infty/B_{\mathrm{EXT}}^{\infty}\,, \tag{36}$$

where B_0 is the downwards uniform inducing field at the surface of the strip, and $2a$ is the width of the strip. The integral in equation (34) would be from $-2a$ to zero. In 1973 *loc. cit.* we demonstrated the *hook over* in the real part of the electric field at a distance C from the edge of a rapidly shelving ocean. If the surface conductivity is a constant $\varkappa_0$ except near the edge, the value of C is given by

$$C = (\omega\mu_0\varkappa_0)^{-1}. \tag{37}$$

(Another value would be needed for a slowly shelving ocean, but for the sake of simplicity we confine our attention to one case only.) Let the tilde distinguish the infinitely conducting model. HEWSON-BROWNE suggested that when the current sheet for the infinitely conducting model is $-2\mu_0^{-1}\tilde{B}_1$, we may assume that the real part of the electric field is given by

$$\operatorname{Re} E = \begin{cases} -2\tilde{B}_1/(\varkappa_0\mu_0) & u \leq -C \\ E_{\mathrm{REAL}} & -C \leq u \leq 0, \end{cases} \tag{38}$$

where E_{REAL} is a constant making Re E continuous. The *coastal enhancement* in the imaginary part of E is modelled by

$$\operatorname{Im} E = \begin{cases} 0 & u \leq -G \\ E_{\mathrm{IMAG}} & -G \leq u \leq 0, \end{cases} \tag{39}$$

where E_{IMAG} and G are constants. Assuming that the integral equation (34) is satisfied on the coastline $u = 0$ and substituting for E from equations (38) and (39) gives sufficient information to evaluate G and E_{IMAG} provided $\varkappa(u)$ is known. HEWSON-BROWNE assumes a uniform surface conductivity $\varkappa_0$ to a distance D from the edge. The conductivity is then assumed to decrease linearly to zero at the edge. Taking the sea water's conductivity to be 4mho m^{-1} and its depth to be 4 km, gives a value for $\varkappa_0$ of $16 \cdot 10^3$ mhos. Taking $\omega = 2\pi/T$, with T measured in seconds, and $\mu_0 = 4\pi \cdot 10^7$, in rationalized m.k.s. units, we obtain $C = 8T$, measured in metres. Thus, with an infinitely conducting half-space at a depth $P/2$ beneath the ocean, we have

$$C \ll P \tag{40}$$

for a range of periods. Moreover, HEWSON-BROWNE also assumes that

$$D < C\,, \tag{41}$$

that is, the oceanic depth shelves to zero over a distance small compared with the scale length for the oceanic current sheet. This is therefore a limited case, suitable only for periods greater than 3.5 h. By the analysis mentioned above, the approximate parameters for a strip of width $2a$ satisfy the equation

$$E_{\mathrm{IMAG}} \approx \frac{E_{\mathrm{REAL}}}{2\pi}\left[\log\left(\frac{a}{C}\right) + 3 + \frac{D}{2C}\log\left(\frac{a}{D}\right) + \frac{3D}{4C}\right], \tag{42}$$

where G is the solution of

$$\frac{G}{a}\left[\log\left(\frac{a}{G}\right) + 1\right] \approx \frac{2\pi C E_{\mathrm{REAL}}}{a E_{\mathrm{IMAG}}} + \frac{D}{2a}\log\left(\frac{a}{D}\right) + \frac{3D}{4a}. \tag{43}$$

For the strip problem, omitting a factor $\mu_0/(\varkappa_0 B_0)$ from the left-hand side the following approximate results were obtained, with the exact results shown in brackets for comparison. For $C = a/20$, $P = a/5$, $D = a/100$.

$$E_{\mathrm{REAL}} = 0.69(0.90) \qquad E_{\mathrm{IMAG}} = 0.91(0.94) .$$

Also, taking $C = a/50$ (equivalent to taking a higher frequency) gives

$$E_{\mathrm{REAL}} = 1.4(1.4) \qquad E_{\mathrm{IMAG}} = 1.3(1.9) .$$

6. Concluding discussion

The developing, and therefore most interesting, regions of theoretical inversion and magnetotellurics appear to be:

(i) Theory and interpretation for large scale magnetometer arrays.

(ii) The production of new methods of solution for general direct problems.

(iii) The stabilization and formulation of new and reliable inversion methods.

(iv) The theory of the qualitative interpretation of observations.

We have not discussed:

(v) The finite difference method of direct solution (*cf.* [19]) as this now seems to be a fairly well established technique, and is being used extensively for modelling. The finite-element method is also currently being used, and we expect papers using it to appear soon. We have also not discussed:

(vi) Advances in formulation, innovation in configurations and analytical solution such as typified by Weaver and Thomson (1972) and the references to Weaver's work contained therein.

Both (v) and (vi) are important, and it is hoped that mathematicians will continue to produce exact solutions for specific configurations. These will be useful building blocks *provided that a principle of superposition* can be developed (*cf.* [58]). Even if such a principle cannot be elucidated, the idea of an "anomalous" region, buried in an inhomogeneous "normal" region, deserves considerable attention.

In (i) and (ii) there is much room for new work. The formulation of general qualitative principles, as in (iv), has hardly been touched.

In (iii), one-dimensional inversion has now reached such a form — in large part due to Parker's promulgation of the Backus—Gilbert view of inversion — that it would be possible as a run up to the next meeting in two years time to hold an inversion competition on some idealized data, chosen to highlight various difficulties occurring in practice.

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МАГНИТОТЕЛЛУРИЧЕСКИЕ ТРЕХМЕРНЫЕ МОДЕЛИРОВАНИЯ И ИНВЕРСИЯ

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РЕЗЮМЕ

Рассматриваются новейшее развитие трехмерных моделирования и инверсии, особенно прогресс за последние годы. Выделяются результаты, достигнутые в области инверсии и измерений в расширенных сетях и из точных решений прямой проблемы тип, являющийся довольно общим для того чтобы его можно было использовать в трехмерной работе по инверсии. В введении кратко излагаются основные уравнения проблемы. Затем, для обоснования дискуссии, определяются понятия «моделирование» и «автоматическая инверсия», как крайние аспекты обсуждаемой территории. Излагаются типичные случаи для маломерных конфигураций: ныне стали доступными новые и эффективные методы в области одномерной инверсии. Авторы касаются и вопроса больших сетей измеренных магнито метром. Это такая область которая полна новых и волнительных вопросов в области исследования глубинного зондирования и аномалий. Но теория разработана только частично: доступными данными еще не решались профили в таком разделении, как на единственной станции с хорошо комбинированными электромагнитными измерениями, все же получились уже значительные результаты. Излагается новый, общий эксперимент для решения прямой проблемы и даются некоторые пометки о краях континентов и о том, как начинают находить правительное направление краевые эффекты вследствие приложенного асимптотического разрешения. Наконец, указывается и на другие, связанные с моделированием, работы, рассматриваются направление развития более широкой области и прогнозы относительно будущего.