Physics of the Earth and Planetary Interiors, 7 (1973) 227–233

THE THEORY OF GEOMAGNETIC INDUCTION

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Accepted for publication February 2, 1973

The theoretical basis of electromagnetic induction methods in geophysics is considered with special reference to the interplay of physical ideas and mathematical arguments. The design of mathematical problems to elucidate observed physical phenomena, and methods of solving them, are discussed. Earth models suitable for various types of investigation are considered. The importance of getting the correct physical interpretation of the solution of a mathematical problem is stressed.

1. Physical ideas and mathematical arguments

In this "Workshop on electromagnetic induction", our aim is to improve our understanding of the induction of earth currents by geomagnetic variations, and thereby improve our techniques for discovering the distribution of conducting material within the earth by measuring and analysing electromagnetic variation fields at the earth's surface. We hope that this will also assist us in our studies of the conductivity of the moon.

This introductory lecture will be concerned only with the theoretical side of this work. In theoretical investigations we have two kinds of tools to help us: physical ideas and mathematical arguments. By the latter I do not mean merely the mathematical solutions of specific problems, but also the mathematical problems to elucidate physical phenomena. There is naturally much interplay between physics and mathematics in our subject. The mathematics is sometimes elaborate and occasionally quite difficult, but I have often found that physical ideas are a great help in suggesting methods of solving the mathematical problems that arise.

An important mathematical part of our subject is concerned with deciding precisely how much we can infer about the conductivity within the earth from a given set of observational data, and with what degree of certainty we can make such inferences. I shall not, however, deal with this so-called "inverse problem", but it will be dealt with by R.C. Bailey in a later review.

I shall confine myself to dealing with the mathematically easier problem of calculating directly the magnetic fields of currents induced in various earth models by varying geomagnetic sources. But let us first remind ourselves that though mathematics is a marvelously powerful tool for supplying the answers in many of our investigations, it can also be a dangerous tool if we do not use it properly. One of the dangers we have to guard against arises from including, for simplicity or convenience, some feature in our mathematical model of the earth, and then drawing inferences about properties of the real earth from our mathematical solution, whereas these deduced properties really stem only from the particular model we have chosen. This probably happens more often than we realise.

2. Basic ideas, equations and units

We are concerned with the flow of electric currents through conducting media. The current flow J will involve some net transport of electric charge, this transport being impelled by the ambient electric field E but impeded by the intrinsic structure of the medium and by its boundaries.

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The effect of the intrinsic structure is usually represented by the specific resistance R, where:

$$E = R\tau \tag{1}$$

In most cases of importance R is independent of J, i.e. the relation (1) is linear. For isotropic media R is a scalar, for non-isotropic media it is a tensor.

The electromotive force, which is part, but not in general the whole of E, usually arises from a varying magnetic field. Also the current J has itself a magnetic field. The relations between the field vectors are of course given by Maxwell's equations:

$$\text{curl}\, \boldsymbol{E} = -\dot{\boldsymbol{B}} = -\mu\dot{\boldsymbol{H}} \tag{2}$$

$$\text{curl}\, \boldsymbol{H} = \mathrm{k}\boldsymbol{J} + \dot{\boldsymbol{D}} \tag{3}$$

where:

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \tag{4}$$

The constant k in eq. 3 has the value 4π in c.g.s. electromagnetic units and the value one in M.K.S. rationalised units. Probably most papers to date on geomagnetic induction have been written in c.g.s. electromagnetic units but many are now written in M.K.S. units, so I have written the basic equations in a form where either system may be used. The really important difference between the two systems is not so much the different values of k, but the difference in the physical relationships between D and E and between B and H, leading to completely different dimensions and numerical values of ϵ and μ in the two systems. In free space, the dimensions and magnitudes of ϵ and μ in c.g.s. and M.K.S. units are as given in Table I. In conducting media the values of ϵ and μ will of course be different but are unlikely to differ much in order of magnitude. It is important to notice the small numerical values of ϵ because this has an important bearing on the interpretation of solutions of certain mathematical problems, as I shall show later.

TABLE I
Dimensions and magnitudes of ϵ_0 and μ_0

	c.g.s. units		M.K.S. units	
	dimensions	magnitude	dimensions	magnitude
ϵ_0	$L^{-2}T^2$	$\frac{10^{-20}}{9}$	$M^{-1}L^{-3}T^2Q^3$	$\frac{10^{-9}}{36\pi}$
μ_0	dimensionless	1	MLQ^{-2}	$4\pi\cdot 10^{-7}$

3. Approximations, simplifications and interpretations

In the usual geomagnetic problem the rate of change of the field is sufficiently slow to permit us to ignore the displacement current $\dot{D}$ compared with the conduction current J in eq. 3. Thus even for conductivities as low as 10^{-17} e.m.u. (10^{-6} M.K.S.) a periodic variation would have to have a period less than 10^{-4} sec to make $\dot{D}$ comparable with J, whereas the periods of the geomagnetic variations that we have to consider are usually greater than 1 sec and frequently much greater.

Neglecting $\dot{D}$, we can often solve the induction problem by taking the curl of eq. 3 and substituting in it the values of J and E given by eq. 1 and 2. Thus, taking for simplicity μ and R to be constants and isotropic, we obtain:

$$\text{curl}\ \boldsymbol{H} = k\,R^{-1}\ \text{curl}\ \boldsymbol{E}\ = k\mu R^{-1}\,\boldsymbol{H} \tag{5}$$

and a similar equation for E. These equations can then be subject to the relevant boundary conditions. Also neglecting D in eq. 3 implies that we can take J as non-divergent to the same order of approximation, and this is frequently useful in obtaining the solution.

Note that in doing this we by-pass eq. 4 completely in calculating the values of H, E and J. But it is important to remark that, when we come to interpreting and explaining the mathematical solution in physical terms, eq. 4 is important, as I shall show later.

4. Global studies

We now consider some of the earth models that have been used for studying the induction of currents in the deeper layers of the earth, where it is assumed that the conductivity κ varies only with the depth. The first simple model was suggested by Schuster in 1889 to explain the part of internal origin of the magnetic daily variations (S). It consisted of a uniformly conducting inner sphere of radius $q\,R_E$ with $q \leqslant 1$, R_E the radius of the earth. In 1919, Chapman showed that $q = 0.96$, $\kappa = 3.6 \cdot 10^{-13}$ e.m.u. gave results that roughly fitted the relations between the fields of external and internal origin, found by spherical harmonic analysis from the observed S-field.

It was shown, however, in 1930 by Chapman and Price that this model was not satisfactory for explaining the part of internal origin of the aperiodic D_{st} field, which required a higher conductivity at greater depths. Hence Lahiri and Price in 1939 introduced a more flexible model in which:

$$\kappa = k\left(\frac{r}{a}\right)^{-m}, a = q^R\mathrm{E}, q \leqslant 1; m \text{ any real number} \tag{6}$$

They found that, if they further elaborated their model by adding an outer shell near the earth's surface of integrated conductivity K, they could account for all the observations available at that date, by either of the distributions d or e shown in Fig. 1, or by any intermediate distribution. Curve d corresponds to $K = 2{\cdot}10^{-6}$ e.m.u. · cm, $q = 1, k = 4{\cdot}10^{-14}$ e.m.u. $m = 37$, and curve e to $K = 5{\cdot}10^{-6}, q = 0.9$ and k and m any values which make $\kappa > 10^{-11}$ The other curves in Fig. 1 indicate other estimates of the conductivity profile derived from studies of other geomagnetic variations, but practically all these studies deal with the first zonal harmonic P_1 only. It will be noted

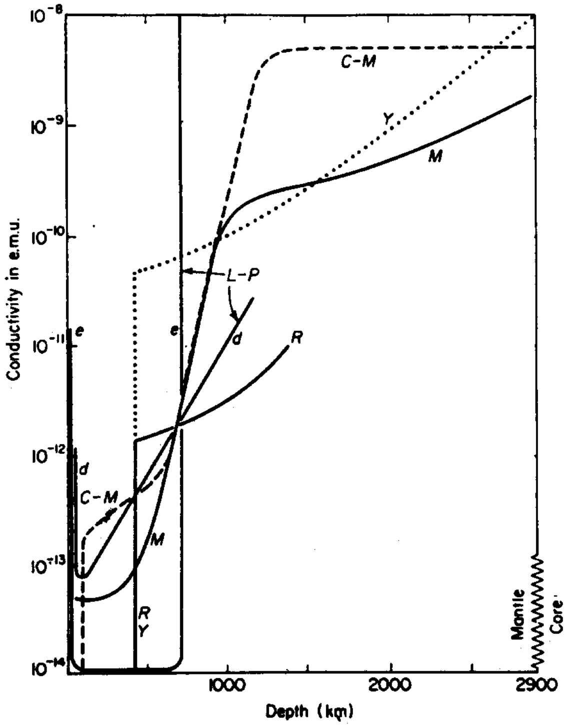

Fig. 1. Conductivity profiles for the earth's mantle suggested by various authors.

that there are considerable differences in these suggested profiles. I have given reasons elsewhere (Price, 1970) for thinking that some of these profiles are unsatisfactory. It seems necessary to emphasize, though it ought not to be, that every suggested profile should be tested to see if it will fit every reliable analysis of all type of variations.

There is, however, one feature common to all but one of the suggested profiles in Fig. 1 which is worth noting. This is the steep rise in conductivity with increasing depth, somewhere between 400 and 800 km. There are good reasons derived from studies of semiconductors at high temperatures and pressures for expecting such a rise, and even possibly a sudden rise due to a phase change at some particular depth. But can we say with certainty that the geomagnetic evidence *alone* can distinguish between these possibilities? Let me remind you that the first earth models, Schuster's and Chapman's, necessarily incorporated a sudden rise in conductivity for purely mathematical simplicity and convenience. How far have these early models influenced our subsequent thinking and our choice of new models? Some investigations apparently claim that the conductivity profiles that they have deduced from geomagnetic induction studies establish the existence of a phase change at a certain depth, but, personally, while I think it probable that a phase change occurs somewhere, I do not believe that the geomagnetic evidence at present available can alone establish this fact incontestably.

Another feature of the above models that needs consideration is the assumption that κ depends on r only. This is obviously not true near the earth's surface, but in the above problems we are interested really in only average values of κ over large regions of dimensions comparable with those of the whole earth. We can therefore treat κ as some smoothed function of spherical polar coordinates (r,θ,φ) except possibly for finite discontinuities at large-scale boundaries like those between continents and oceans. This average κ will smooth out the immediately local variations of κ. Near the earth's surface this average κ will not be horizontally uniform, i.e. it will depend on θ and φ as well as r, and some investigators consider that this may also be true in the upper part of the mantle. But at depths greater than 100 km or so it is likely that κ will depend much more on r than on θ or φ because of the temperature and pressure effects. This is the justification for using the above models for slow variations and correspondingly deep probing.

There are of course the other global problems that relate to the crust, in which the horizontal variation of κ is of first importance, much as in the different conductivities of seawater and continental rocks. In some of these problems it is useful to consider the theory of currents induced in non-uniform thin shells. The basic theory and some methods of solving certain problems were given by Price in 1949. These methods included two iteration methods that afford a good illustration of what I said earlier about using physical ideas to suggest procedures for solving the relevant mathematical problems. The current τ induced in a thin non-uniform shell by periodic variations of period $2\pi/\omega$ can be derived from a stream function ψ, because J is (practically) non-divergent. Price showed that ψ satisfies the equation:

$$R \ \text{div grad}\ \psi + \text{grad}\ R \cdot \text{grad}\ \psi = i\omega(N^e + N^i) \qquad (7)$$

where R is the variable sheet resistance and N^e, N^i are the normal components of the induced and inducing fields, respectively. Since the equation is linear, ψ can be regarded as made up of: (1) ψ^e due directly to N^e, and (2) ψ^i due to N^i. The latter represents the self-induction effect and when this is small we can get a first approximation ψ_1 to ψ by solving eq. 7 with $N^i = 0$. The magnetic field of these currents ψ_1 can then be calculated to get a first approximation N_1^i to N^i, and N_1^i can then be used in eq. 7 to get a second approximation ψ_2 to ψ. This procedure can obviously be repeated to get successive approximations. The iteration method thus suggested by physical considerations is found to be convergent for sufficiently small values of $\omega L/R$, where L is a length determined by the dimensions of the conductor and the spatial scale of the inducing field.

On the other hand when $\omega L/R$ is large everywhere, the normal component of the induced field will nearly cancel with that of the inducing field, so that we can start with $N_1^i = -N^e$ as a first approximation. We then calculate the current function ψ_1 that will give rise to N_1^i, and substitute this in eq. 7 to get a second approximation N_2^i and so on, thus giving a second iteration method.

Important applications and extensions of these methods have been made by Ashour, Hobbs. Rikitake, Hutson and others. Two iterations methods have been recently described by Hutson et al. (1972) in which successive approximations to the actual current density J are obtained in terms of the vector potential of the magnetic field and the scalar potential of the associated electric field. Except for problems having axial symmetry their methods become rather complicated, but they can be transformed to give iterative calculations for ψ. When this is done, their first method is seen to be identical with Price's first method, and with regard to their second method, they have recently described it as one in which "essentially the range of Price's first method is extended" This extension is particularly valuable when the distribution of R is discontinuous.

5. Local problems and flat earth models

Apart from the above global problems, there are other induction problems relating to strictly limited areas of the earth's surface. In these we can ignore the sphericity of the earth and consider various flat-earth models with horizontal strata having either uniform or non-uniform conductivities. The simplest such model was that used by Cagniard in 1953 to develop his "magnetotelluric method" of conductivity sounding. This model consisted of horizontal strata extending to infinity in the x- and y-directions, the conductivity being a function of the depth z only, He defined the "apparent resistivity" for such a model as:

$$R_a = 2T \left| \frac{E_x}{H_y} \right|, \text{where } T = \frac{2\pi}{\omega} \qquad (8)$$

Actually this formula is derived from the skin effect formula for alternating currents in a uniform half-space conductor, and is not directly concerned with the source field of the magnetic variations.

When the resistance of the half-space conductor is uniform the expression on the right of eq. 8 is a constant for all T. When the conductivity is a function of z, the expression is used to define the apparent resistivity for the particular values of T. Since the currents penetrate deeper for greater T, the way in which R_a varies with increasing T gives an indication of the conductivity profile. The accuracy of this indication will depend on how the Cagniard model fits the circumstances of the particular geophysical investigation being undertaken. When we have a sufficiently uniform terrain, the method is a convenient and valuable one for estimating the conductivity profile down to moderate depths. For great depths, however, even when the horizontally homogeneous strata are extensive, it may be necessary to consider the nature and dimensions of the source field, discussed by Wait in 1954 and in more detail by Price in 1962. In this case the theory does not really differ basically from that of geomagnetic deep sounding in which the vertical component of H is used, and E is not used.

Other problems arise when the strata are not horizontally uniform over extensive regions, but gradual or sudden changes of conductivity occur. We now consider these briefly.

6. Local anomalies of magnetic variations

There are two important types of induction effects arising in geomagnetic variations that are due to the non-uniformity of the earth's crust. One is concerned with coastal and island effects arising from currents induced in the sea, and we have briefly referred to this in § 4. The other is concerned with land areas (not necessarily near the sea) where the normal geomagnetic variations such as the daily variations and various disturbance variations like bays and pulsations are overlain with distinctive local features. In such areas the vertical magnetic variation, in particular, is often decidedly anomalous. Rikitake first called attention in 1952 to such an area in Japan, and in the last twenty years many others have been found in all parts of the world.

The study of these anomalous areas has become a very important part of our subject since it may reveal considerable information about subterranean geological features, but it was some time before really satisfactory interpretations of the anomalous observations could be obtained because of the difficulty of solving the mathematical problems involved. I attempted to describe the general nature of these problems in 1962, and I then said of them "These may prove laborious to solve, but it is to be hoped that eventually a library of solutions of relevant problems will be available. This would help the task of interpreting many geomagnetic variation phenomena."

During the last decade, the advent of large modern computers has made the solution of many of these problems possible, and excellent progress has been made in designing and solving mathematical problems to elucidate the phenomena. Several mathematical and computer techniques have been developed for this purpose. I need not describe any of these here, but I should like to refer to the solution of one particular problem obtained by Jones and Price (1970), because it raises the question of the physical interpretation of the mathematical solution, which has led to some controversy, and will illustrate some of the points I have made earlier.

7. Induced currents that go astray

The problem concerns the perturbation of alternating currents flowing in a half-space conductor occupying $z \geqslant 0$, by a vertical discontinuity of conductivity at the plane $y = 0$. The current flow at an infinite distance from the origin will be horizontal and distributed according to the usual skin effect. This current flow at infinity can be separated into two components, one flowing in the x-direction and one in the y-direction and the total perturbation effect can be calculated by superposing the effects on each of these components. We consider the case when the current flow at infinity is in the y-direction, so that the currents will impinge on the plane of discontinuity $y = 0$. This will correspond to H-polarisation, i.e. the magnetic field lines will all be parallel to the x-axis. In the neighbourhood of the discontinuity the currents will be disturbed but will always flow in planes parallel to $x = 0$. The differential equations and boundary conditions of the problem were solved numerically. The solution shows some remarkable effects of the discontinuity on the surface values of E, but H outside the conductor is uneffected by the discontinuity. These surface field results agree in character with those found by d'Erceville and Kunetz (1962) from an analytic solution. However, one method of solution also enabled us to find the field and current flow inside the conductor, and Fig. 2 shows the lines of current flow for one particular epoch during the period of oscillation. For the epoch shown the flow is from left to right, from the conductor κ_1 to the conductor κ_2 where $\kappa_1 = 10\,\kappa_2$. It will be noted that our solution shows a significant downward turn of the current lines as they approach the boundary and a sharp refraction at the boundary. What causes the currents to stray from their straight and horizontal paths that they follow at distances well away from the discontinuity in κ? We say that the cause of this – the villain of the piece! – is an alternating surface charge being continually placed on the boundary by the impinging current as shown in Fig. 1. It is this suggestion that has aroused some controversy. We argue that the electric field of this surface charge, which is quite minute if measured in coulombs, has nevertheless an electric field comparable with the other electromotive forces in the problem. The normal component of this electric field opposes the extraneous field E_0 in the better conductor κ_1 and adds to it in the other conductor κ_2, thus effectively (but theoretically not quite absolutely) equalising the normal components of current flow across the interface. Also the surface charge is greater at the upper end of the interface because the impinging current is most intense there due to the skin effect of the horizontal upper surface. This produces a net electric force E_z downwards, which bends the currents down as in the figure. Also, if we take for simplicity the dielectric constants ϵ to be the same in the two conductors, we have $E_{1n} = -E_{2n} = E_n$ say, so that the resultant horizontal fields in the two conductors are $E_0 - E_n$ and $E_0 + E_n$, while the vertical fields are the same. Hence the lines of electric force, and therefore also the current lines, are refracted at the interface as shown in the figure.

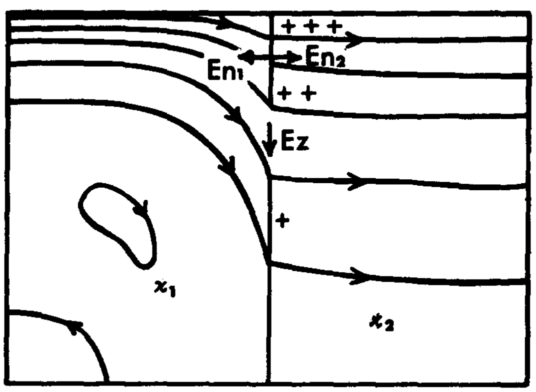

Fig. 2. Current lines and surface charge distribution near a vertical plane of discontinuity of conductivity.

Now the above conclusions all rest on the argument that a surface distribution of charge is brought into existence at the interface. This, however, has been queried and indeed denied by some investigators (see for example Jones and Price, 1972). This is not really surprising because the equations used to solve the problems, i.e. eq. 1, 2 and 3 together with the boundary conditions, contain no explicit reference to any charge distribution, and when the displacement current is ignored, eq. 2 shows that the current becomes nondivergent. Evidently, if the villain *is* a surface charge, he is well hidden! We can now discover how he is hidden. Firstly, the electric field of a charge distribution is a gradient vector and the curl of a gradient vector is zero. Hence by taking the curl of E in eq. 2 we have already hidden any part of E due to a charge distribution. But we still have the difficulty that there can be no charge accumulating anywhere because J is non-divergent. This, however, arises from our assumption that the displacement current can be ignored, compared with the conduction current, which is certainly true for the time-rates of change involved. But our assumption that we can ignore $\dot{D}$ must of course imply that we can ignore any current *of the same order as* D. Now if a surface charge of density ρ is built up on the interface, we have:

$$D_{1n} + D_{2n} = \rho \quad (9)$$

and therefore the current extracted from τ_n to build up ρ is:

$$J_{1n} - J_{2n} = \dot{r} = \dot{D}_{1n} + \dot{D}_{2n} \quad (10)$$

which is of the same order of magnitude as the displacement current and can therefore be ignored. Nevertheless, the electric field of the surface charge is *not* negligible, because, for example, $E_{1n} = \epsilon_1^{-1} D_{1n}$, and the very small values of ϵ mentioned in § 2 ensure that the electric field of the charge itself is comparable with that due to the applied e.m.f. driving the currents, in spite of the fact that the magnetic field of the extracted current to build it up is negligible.

One further point that I would like to emphasize is that the current produced by the electric field of the charge distribution is a true *conduction* current, proportional to the conductivity of the particular conductor. It is *not* a displacement current as some investigators have suggested. This misconception has probably arisen from eq. 10 which shows that the current required to *setup* the charge distribution can be expressed in terms of displacement currents, but the current *arising* from the electric field of this distribution is of a quite different character.

The reader may perhaps feel that I have rather laboured the above discussion of interpretations unnecessarily, but I have found so many misconceptions about these physical ideas that it seems to me desirable to clarify the physical picture as much as possible.

8. The moon and beyond

The induction methods developed for probing the earth's conductivity are now, of course, being applied to the moon, and will probably soon be applied to the planets. It is interesting to note that already a number of quite different conducting profiles have been suggested for explaining various features of the Apollo magnetometer results. The history of the investigations of the earth's conductivity would lead us to expect this, and we can probably learn much by studying that history. Undoubtedly our aim must be to ultimately find a model that will fit *all* the available experimental data, and then to decide how accurately and with what degree of certainty, we can define that model. I will not discuss this further, but I would like to end with a "Cautionary Tale". In the preface (I think it was, but it's a long time ago since I read it) to one of Heaviside's volumes on *Electromagnetic Theory*, there is a tale about a mathematician who went slightly mad, and was greatly affected by the moon. He was so madly obsessed with the moon, that he made a beautiful model of it, and then he became convinced that his model was the real moon, and the thing in the sky merely a figment of the imagination! Need I say more than "Mathematicians beware of being seduced by your beautiful models!"

References

(References to many of the topics discussed are given in the more detailed reviews which follow. Here only a few specially relevant references are given.)

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