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INDUCTION IN LATERALLY NON-UNIFORM CONDUCTORS: THEORY AND NUMERICAL MODELS

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The study of electromagnetic induction in laterally non-uniform conductors is briefly reviewed. The two-dimensional perturbation problem is considered and the two polarization cases which arise from Maxwell's equations are discussed. Techniques for the solution of the equations for laterally non-uniform conductors are discussed with emphasis on numerical methods.

1. Introduction

In a short paper entitled "A note on the interpretation of magnetic variations and magnetotelluric data" which was included with other valuable papers on electromagnetic induction in the earth communicated to the I.U.G.G. Symposium on the Upper Mantle Project at Berkeley in 1963, and edited by Rikitake (1964a), Price (1964) has outlined the nature of methods of investigation of the electrically conducting material within the earth and the mathematical problems encountered. He points out that the mathematical problems to be considered fall into two groups, depending on whether we are concerned with global properties, that is, properties of the earth as a whole, which he terms "global problems", or with local properties such as the electrical conductivity of localized regions near the surface of the earth which he terms "local problems".

In the local problems we are concerned with quite limited regions and are interested in variations of σ that occur over distances of the order of 100 km or less. In these problems the earth is treated as a semi-infinite or thick plate conductor with variable distribution of conductivity, and we are interested in only a limited region of this conductor. Price (1964) points out that the kind of problem we need to consider in this connection is not strictly a problem of evaluating the currents induced by a given varying magnetic field in a given heterogeneous conductor, but rather that of determining the local perturbations of a given alternating system of induced currents by given abrupt changes of conductivity. Price describes the mathematical problem for such disturbed skin effect problems as:

> "Using Cartesian co-ordinates (x,y,z) with the z-axis vertically downwards, a non-uniform conductor occupies the half-space $z > 0$. Near the origin the conductivity is a function (not necessarily continuous) of (x, y, z), but at great distances from the origin it is a function of z only. A given alternating e.m.f. impels currents near the surface of the conductor. To determine the distribution and surface field of these currents."

The given e.m.f. represents the electric field arising from a varying external magnetic field.

It should be emphasized that for the local perturbation problem the plane-earth approximation is used. The first study of induction in a plane earth was for a semi-infinite plane earth with uniform conductivity and was given in a classic paper by Price (1950). Following this, uniformly layered earth models were considered by Tikhonov (1950), Cagniard (1953) (from whose paper the well known expression for the so-called Cagniard apparent resistivity was derived) and Wait (1954, 1962), as well as by Price (1962, 1965) and others. In the initial work in his 1950 paper, Price presented a general theory considering any inducing field which was assumed known. Cagniard (1953), in his development of the theory leading to the magneto-telluric method, assumed a spatially uniform source of infinite extent and sinusoidally periodic in time, as well as a semi-infinite uniformly stratified earth with a plane boundary. The paper cited above by Wait (1954), which is followed in the journal by a record of the interesting correspondence between Dr. Wait and Professor Cagniard, discusses these assumptions. The two papers by Price (1962, 1965), mentioned above, also consider the assumption concerning the dimension of the source field and its effect on the calculation of the Cagniard apparent resistivity. Many other papers have appeared in which induction in a semi-infinite conducting half-space with a plane boundary is considered, including a recent paper by Weaver (1971), who has developed the theory in terms of one scalar component of the magnetic Hertz vector.

In 1962 two papers appeared in *Geophysics*, one by d'Erceville and Kunetz (1962) and the other by Rankin (1962), in which vertical discontinuities in conductivity were discussed. During the last decade there has been increasing attention given to the problem of lateral non-uniform distributions of conductivity.

Lateral inhomogeneities in conductivity are of considerable interest, and reports of geomagnetic anomalies associated with such inhomogeneities have appeared frequently in the literature. In a report to the Upper Mantle Project Symposium at Berkeley in 1963, and later in a very comprehensive report, Schmucker (1964, 1970) has classified the lateral inhomogeneities into three types of anomalies: (1) surface anomalies, which are due to superficial conductivity variations above the crystalline basement; (2) intermediate anomalies which are connected with insulated conductors in the high-resistivity zone of the earth's crust and uppermost mantle; and (3) deep anomalies, which reflect conductivity imbalances in the upper mantle.

Recently, in a talk at the Ninth International Symposium on Geophysical Theory and Computers, Gough (1972) considered five more specific types of anomalies associated with geomagnetic deep-sounding studies. Furthermore, he has indicated that there exist, under certain conditions, variable transmission or "vartran" anomalies which are due to refraction/absorption effects in which conductive bodies are seen in transmission fields.

Included in Schmucker's surface anomaly type is the lateral discontinuity caused by the oceanic–continental interface. The abrupt discontinuity in conductivity between land and sea results in a perturbation of the induced currents due to externally varying magnetic fields. These perturbations are apparent in the data from magnetic measuring stations, and appear principally as a dependence of the amplitude of the vertical magnetic variations on the distance from the continental shelf. This coastal effect has been examined in detail by Schmucker (1964) for variations near the coast of California. Anomalous variations in the vertical magnetic component in coastal regions have been observed in many countries. Also, Parkinson (1959, 1962, 1964) has shown that at many stations there is a preferred plane for the vector changes of the geomagnetic field. That is, there is a strong tendency for the vertical component to increase or decrease when the change in the horizontal component is along a particular direction.

These coastal effects are usually attributed to the effect of currents induced in the sea, but some results of theoretical studies tend to suggest that currents induced in the oceans themselves are not the full cause of the effects (Parkinson, 1964; Cox et al., 1970; Bullard and Parker, 1970). It appears that in the coastal effect it is difficult to separate the effects due to induced currents in the ocean from the effects due to the differences in the mantle under the oceans and the continents.

A second effect associated with electric currents in the ocean is the "island effect". Magnetic variations have been made on floating ice islands (Zhigalov, 1960; as well as others) and when such measurements are compared with observations on oceanic islands (Mason, 1963; Klein, 1971) distinct differences are observed. The differences are attributed to the fact that islands rising from the ocean beds interrupt the current flow, while ice islands do not. The effect is described by Price (1967).

Other geomagnetic anomalies not necessarily associated with the oceans have been observed by workers in various locations. Since the early 1950's the electrical conductivity structure of the earth has been studied by using networks of magnetometers and reported on by many workers. Porath and Dziewonski (1971a) have given a useful review of crustal resistivity anomalies from magnetic deep-sounding studies. Particularly productive operations of large two-dimensional arrays have been made by two groups in North America, one at the University of Texas at Dallas, and the other at the University of Alberta in Edmonton (Gough and Reitzel, 1969; Reitzel et al., 1970; Porath et al., 1970; Porath and Dziewonski, 1971b; Porath and Gough, 1971; Porath et al., 1971; Camfield et al., 1971;

and Gough and Camfield, 1972). Through the operation of these arrays, mantle conductive structures as well as surface conductivity anomalies have been mapped. Also, Lilley et al. (1971) and Lilley and Bennett (1972) have recently operated arrays in Australia and D.I. Gough (personal communication, 1972) used arrays in South Africa from which much information should be forthcoming. Edwards et al. (1971) have reported on the results of geomagnetic variation measurements from arrays of magnetometer stations throughout the British Isles.

Various techniques have been used to model conductivity anomalies and so lead to a reasonable inversion of the data. Analogue models have been used by Rankin et al. (1965), Roden (1964), Dosso (1966a,b,c; 1969), Dosso and Jacobs (1968), Thomson and Dosso (1971), and Ogunade and Dosso (1971), to investigate laterally non-uniform structures. Porath et al. (1970) have separated the fields into normal and anomalous parts and used numerical calculations of model studies for inversion. Schmucker (1964) also considers the normal and anomalous parts of the observed variations and assumes that the anomalous variations are of internal origin.

2. The two-dimensional model

2.1. The differential equations

The model is that of a semi-infinite conductor with a plane boundary occupying the region $z > 0$ and which may have regions of different conductivity.

The problem is basically that of solving Maxwell's equations in the various regions with suitable boundary conditions. The field is taken to be an oscillating one with period $2\pi/\omega$ which is sufficiently long so that displacement currents may be ignored (Price 1950, 1967). Also, the magnetic permeability is taken as unity. The equations, in electromagnetic units are then:

$$\nabla \times \boldsymbol{H} = 4\pi\sigma \boldsymbol{E} \qquad (1)$$

and: $$\nabla \times \boldsymbol{E} = -i\omega \boldsymbol{H} \qquad (2)$$

where the time factor $\exp(i\omega t)$ is understood in all field quantities, and σ is the conductivity appropriate to each region.

In the two-dimensional problem it is assumed that all quantities are independent of one direction (in this case the x-direction) and only variations in the plane perpendicular to this direction are considered. Since all quantities are independent of x, eq.1 and 2 reduce to:

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 4\pi\sigma E_x \qquad (3a)$$

$$\frac{\partial H_x}{\partial z} = 4\pi\sigma E_y \qquad (3b)$$

$$-\frac{\partial H_x}{\partial y} = 4\pi\sigma E_z \qquad (3c)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega H_x \qquad (4a)$$

$$\frac{\partial E_x}{\partial z} = -i\omega H_y \qquad (4b)$$

$$-\frac{\partial E_x}{\partial y} = -i\omega H_z \qquad (4c)$$

In these six equations, E_x, H_y and H_z are involved only in eq.3a, 4b, 4c, while only H_x, E_y and E_z are involved in eq.3b, 3c, 4a. As a result, we can solve these two separate sets of equations independently. The first set corresponds to what may be called E-polarization. In this set, by eliminating H_y and H_z, we get:

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = i\eta^2 E_x \qquad (5)$$

as the equation to be solved. The second set corresponds to H-polarization, and by eliminating E_y and E_z we obtain:

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = i\eta^2 H_x \qquad (6)$$

as the equation to be solved in the various regions. In these equations:

$$\eta^2 = 4\pi\sigma\omega \qquad (7)$$

These equations may be solved for either E_x or H_x depending upon which case is being considered, and then the other components for that case may be obtained from eq.3b and 3c or eq.4b and 4c.

2.2. The boundary conditions

The boundary conditions for the two-dimensional problem are considered in detail by Jones and Price (1970). At the interface between conductive media and also at the surface $z = 0$, we have the boundary conditions that: (1) all components of $\mathbf{H}$ are continuous, and (2) the tangential components of E are continuous. Also, the normal component of current density must be continuous across conductivity discontinuities and zero across $z = 0$, which implies that $E_z = 0$ inside the conductor at $z = 0$.

There are also conditions to be satisfied at $z = \pm\infty$ and $y = \pm\infty$. The basic condition in this respect is that the boundaries must be placed far enough from any lateral discontinuity so that the fields may be considered uniform there. Also, there is a difference between the H and E-polarization cases.

In the region outside the conductor, where $\sigma = 0$, H_x is independent of y and z for the H-polarization case. This implies that $\mathbf{H}$ is uniform throughout this region and the magnetic field immediately above the surface of the conductor is not affected at all by the abrupt changes of conductivity within the conductor. Along the surface of the conducting region H_x remains constant, and so this provides a boundary condition for this case.

In the E-polarization case, the $\mathbf{E}$-field in the region $z < 0$ must be considered. The half-space conductor may be considered as the limit of a spherical conductor as the radius tends to infinity. Price (1950) has shown that the ratio of the tangential components of the induced and inducing fields is independent of the conductivity when the radius tends to infinity, and assuming that the inducing field is of the same intensity and form over the entire composite conductor, then the correct boundary condition is that the total surface H_y is the same at $y \to +\infty$ and $y \to -\infty$ as well as at some height $z = -|h|$ which is at sufficient height so that the local perturbation in $\mathbf{H}$ is negligible there.

Within the conductor it is assumed that the boundaries are far enough from any lateral discontinuities so that at $y \to \pm\infty$ the fields behave like those for a uniform or horizontally layered conductor. Also, the fields tend to zero as $z \to +\infty$.

Schmucker (1971b) has taken a different approach to the boundary conditions for the E-polarization case. He assumes that the anomalous field is of totally internal origin and expresses the boundary condition at the surface of the conducting region and the lower horizontal boundary in terms of integral relations involving the horizontal and vertical magnetic components.

3. Analytical solutions for laterally non-uniform conductors

In their study of electromagnetic induction in a semi-infinite conductor with a plane boundary, d'Erceville and Kunetz (1962) considered a model with two media of different resistivities in contact along a vertical plane overlying a horizontal basement that was taken as being either infinitely resistive or infinitely conductive or at infinite depth. In their analysis, d'Erceville and Kunetz obtained an exact mathematical solution for the H-polarization case in which the magnetic field is parallel to the strike, but only briefly, at the end of their paper, mentioned the E-polarization case.

Rankin (1962) applied the method of d'Erceville and Kunetz to the case of a dike of infinite length but finite depth and in which the magnetic field is parallel to the dike. He did not consider the E-polarization case.

In the year following the work by d'Erceville and Kunetz and by Rankin, a very interesting paper which has stimulated much work in the field, by Weaver (1963) appeared. Weaver's model was that of a semi-infinite conductor with a plane boundary consisting of two quarter-spaces of different finite conductivity. He considered both the H-polarization and E-polarization cases. In the E-polarization case it was necessary for him to use an approximate boundary condition requiring the horizontal magnetic field at the surface of the conductor to be constant. From his E-polarization solution, Weaver was able to show the increase in amplitude of the magnetic component normal to the surface of the conductor as the region of the discontinuity is approached, and related this to the enhancement of the vertical magnetic component of magnetic variations observed at coastal stations.

Weaver and Thomson (1972) have recently elaborated on Weaver's 1963 work by using a perturbation technique proposed by Mann (1970) and they have avoided in the E-polarization case the use of the earlier approximate boundary condition. Weaver and Thomson (1972) have obtained approximate solutions for a periodic line current above a non-uniform earth and have found expressions for the field for the case when the height and magnitude of the line current approach infinity in such a way that the inducing field near the earth becomes uniform and finite.

Geyer (1970) has used a similar perturbation technique to investigate the electromagnetic anomalies over several types of subsurface structures, and more recently (Geyer, 1972) has extended this to a dipping contact. Previous to this, Yukutake (1967) considered induction in a conductor bounded by an inclined interface with a small angle of tilt by a successive approximation method which used repeated reflections of electromagnetic energy between the ground surface and the tilted boundary.

Recently, Treumann (1970a,b,c) has considered induction in non-uniform plates of finite thickness, and has been able to obtain approximate solutions for the field at the surface of the plates when the external inducing field is uniform. Also, Weidelt (1971) has studied induction in two adjacent half sheets with different uniform conductivities. Furthermore, Schmucker (1971a) has used convolution integrals to investigate the induction in a model with a non-uniform surface layer above a layered substratum.

4. Numerical methods

There are several methods of solution of the perturbation problem which may be termed pure numerical methods, that is, methods in which the equations are evaluated from the beginning by numerical means.

The analytical methods generally give solutions for only specialized cases. Anomalous conductivity structures of any shape may occur naturally, and so it is necessary to consider methods to deal with inhomogeneities of arbitrary shape. As a result, numerical methods have been developed to deal with general two-dimensional problems and in this section three such methods will be discussed. One of the methods, that used by Jones and Price (1970), Jones and Pascoe (1971) and Pascoe and Jones (1972), has now been extended by Jones and Pascoe (1972) and Lines and Jones (1973a, 1973b) to consider three-dimensional models.

In the pure numerical methods, the approach taken is to consider any arbitrary conductivity distribution within a particular region that can be represented by a two-dimensional model, and then impose a mesh of grid points over this region with the boundaries of the mesh taken at far enough distances from any lateral discontinuities so that the boundary conditions are satisfied. In the simplest application, the mesh lines joining the grid points are taken as boundaries of conductive units or cells, which when placed together may represent a conductive structure with various regions of different conductivity.

4.1. The transmission line analogy

The transmission line analogy arises from the similarity in form between Maxwell's equations governing the orthogonal components of $\boldsymbol{E}$ and $\boldsymbol{H}$, and the transmission line equations governing current and voltage on a transmission line, or, since we are considering a two-dimensional problem, over a transmission surface with current flowing in two directions. This analogy, first applied by Dulaney and Madden (1962) to such problems as we are considering, has been used by many authors (including Madden and Thompson, 1965; Madden and Swift, 1969; Swift, 1967, 1971; Wright, 1969, 1970; and others).

In the transmission line analogy method for the H-polarization case the electric fields in Maxwell's equations are represented by currents in the branches of the mesh, while the magnetic field is represented by the voltage at the node being considered. Also, the quantity $4\pi\sigma$ in the equations of section 2 is represented by the impedance Z, while the quantity $i\omega$ is represented by the admittance Y. That is:

$$I_y \langle=\rangle E_z \tag{8}$$

$$I_z \langle=\rangle -E_y \tag{9}$$

$$V \langle=\rangle H_x \tag{10}$$

$$Z \langle=\rangle 4\pi\sigma \tag{11}$$

$$Y \langle=\rangle i\omega \tag{12}$$

so that in terms of this correspondence the equations for the H-polarization case as given in section 2 may be written:

$$\frac{\partial V}{\partial y} = -Z I_y \tag{13}$$

$$\frac{\partial V}{\partial z} = -Z I_z \tag{14}$$

$$\frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z} = -Y V \tag{15}$$

Eq.13 and 14 may be written:

$$\nabla V = -Z I \tag{16}$$

and eq.15 as

$$\mathrm{div}\, I = -Y V \tag{17}$$

Eq.16 and 17 are the two-dimensional transmission equations.

If the mesh is assumed to be composed of unit cells that are electrically homogeneous, Kirchoff's law of current continuity may be written for each node and a set of equations is obtained, the solution of which gives the value of the voltage (corresponding to H_x in this case) at each node.

For the *E*-polarization case a similar approach is taken and the transmission surface equations (16 and 17) are again obtained. These are solved for the voltage at each node which corresponds this time to the value of E_x at that node.

It is clear that the proper boundary conditions must be applied to the impedance network and these are obtained by considering the layered conductor at the boundary to be a one-dimensional transmission line. Solutions to the equations are obtained by inversion of the matrices and the methods are discussed by Wright (1969) and others.

4.2. The finite-element method

The use of the finite-element method when applied to engineering problems is described by Zienkiewicz (1971) and this method has been used in electromagnetic modelling problems related to geophysical conducting structures by Coggan (1971), Ryu (1972) and Reddy and Rankin (1972).

In this method, as in the other pure numerical methods, the region of interest is sub-divided into a mesh of elements. In this case, the principle used is that electromagnetic fields behave so as to minimize the total energy. Thus, the energy in the electromagnetic field within each element is considered and the total energy is numerically minimized. Coggan (1971) outlines the method in detail. The total energy is expressed as an integral over space of several energy densities. Again, a set of equations with field values at the nodal points of a mesh as unknowns is obtained, and this set is solved by matrix inversion.

4.3. The finite-difference method

The finite-difference technique was first used by Neves (1957) to study electromagnetic induction in two-dimensional structures. Also, Latka (1966) and Patrick and Bostick (1969) have used this method. Jones and Price (1969, 1970, 1971a,b) have employed this method for studies of various two-dimensional conductivity distributions and Jones and Pascoe (1971) and Pascoe and Jones (1972) have given a general computer program for the solution of the local perturbation problem. Jones (1972) has recently extended this to calculate only the perturbation field associated with the discontinuities.

As developed by Jones and Price (1969, 1970), and programmed by Jones and Pascoe (1971), the method of solution involves the solution of the appropriate finite-difference equations over a mesh of grid points by the Gauss-Seidel iterative method (Smith, 1969). The equations (5 and 6) to be solved in all regions for both the *E*-polarization and *H*-polarization cases are of the form:

$$\nabla^2 F = i\eta^2 F \tag{18}$$

where $\eta^2 = 4\pi\sigma\omega$ and F is either E_x or H_x, depending upon the case we are considering. If we let $F = f + ig$ then:

$$\nabla^2 f + i\nabla^2 g = i\eta^2 f - \eta^2 g \tag{19}$$

and equating real and imaginary parts we obtain:

$$\nabla^2 f = -\eta^2 g \tag{20}$$

$$\nabla^2 g = \eta^2 f \tag{21}$$

A mesh of grid points is superimposed over the region of interest and this pair of equations is written in finite-difference form and solved simultaneously at each point by the iteration scheme.

It should be emphasized that in all these numerical methods continuous functions are represented by values at discrete points over a mesh of finite dimensions. Much care must be taken in the choice of mesh size, and in the placing of the boundaries of the grid with respect to conductivity discontinuities.

Also, it should be pointed out that in all the numerical methods some functional form (usually linear) is assumed for the fields between grid points. This implies that the calculations of the initial component as well as the other components are approximate and the accuracy depends on the grid size.

5. Some results from numerical models

5.1. Characteristics of the perturbation fields

Many of the results obtained from numerical methods have been with direct reference to the magnetotelluric method and have mainly considered surface values of the field components. However, Jones and Price (1970) plotted contours of field values over the whole region of interest as a function of time and examples are given in their paper. For the *H*-polarization case they plotted contour lines of constant H_x (≡ lines of force of the *E*-field) and observed strong refraction of the lines of current flow at discontinuities as well as current vortices which migrate downward in the conductor and decay with time. They attributed the refraction of the current lines to the existence of a minute varying electric charge distribution on the interfaces, whose magnetic effects are of the same order as the displacement currents and so are neglected, but whose electric field is of the same order of magnitude as that of the other electromotive forces involved. This interpretation has been discussed further by Hermance (1972) and Price and Jones (1972).

In the *H*-polarization case, although the distribution of the currents and the fields within the conductor is greatly perturbed by the existence of the discontinuity, the magnetic field outside the conductor remains uniform. The total current system may then be thought of as consisting of two parts: (1) sheets of currents flowing parallel to the surface and contributing to the uniform field outside; and (2) a toroidal current system whose magnetic field is contained entirely within the conductor.

For the *E*-polarization, Jones and Price (1970) plotted contours of equal E_x (≡ lines of force of the *H*-field) and observed wedges of high current density which are formed during each cycle. Such current concentrations are continually formed and move downward in the conductor and decay with time. They emphasized that in this case the abrupt change in continuity at $y = 0$ has two distinct effects. One is the local perturbation in the electromagnetic field near $y = 0$, which will decrease with increasing negative z because it is due to the local concentrations of current, and the other is the effect of the current distribution and field because of the different conductivities at infinity in the positive and negative y-directions.

5.2. Apparent-resistivity calculations and lateral conductivity anomalies

Although Cagniard (1953) derived the expression for apparent resistivity:

$$\rho_A = 2T\left(\frac{E_x}{H_y}\right)^2 \qquad (22)$$

assuming both a uniform source of infinite extent, and a horizontally stratified earth, the calculation of apparent resistivity is still made by those investigating non-uniform sources as well as for situations in which the condition of horizontal stratification is not satisfied. In fact, although the conditions assumed for the original definition set down by Cagniard may not necessarily be valid in a certain circumstance, it is still possible to calculate the apparent resistivity and use it as a comparative parameter. However, from the results obtained in the calculations (see for example, Hibbs and Jones, 1972a), it is clear that apparent-resistivity calculations must be treated with care in cases with lateral discontinuities.

6. Source considerations in the induction problem

Thus far all comments concerning the theory of numerical methods and their results have pertained to situations in which the inducing field is uniform and of infinite extent. In his original discussion of the magnetotelluric method, Cagniard (1953) assumed such a source above a layered conductor. As mentioned before, Wait (1954) has criticized this assumption. Also, Price (1962) criticized this assumption and has shown that the simple Cagniard formulas on which the magnetotelluric methods are based require modification to take into account the distribution of the ionospheric field. This point is important since in various regions,

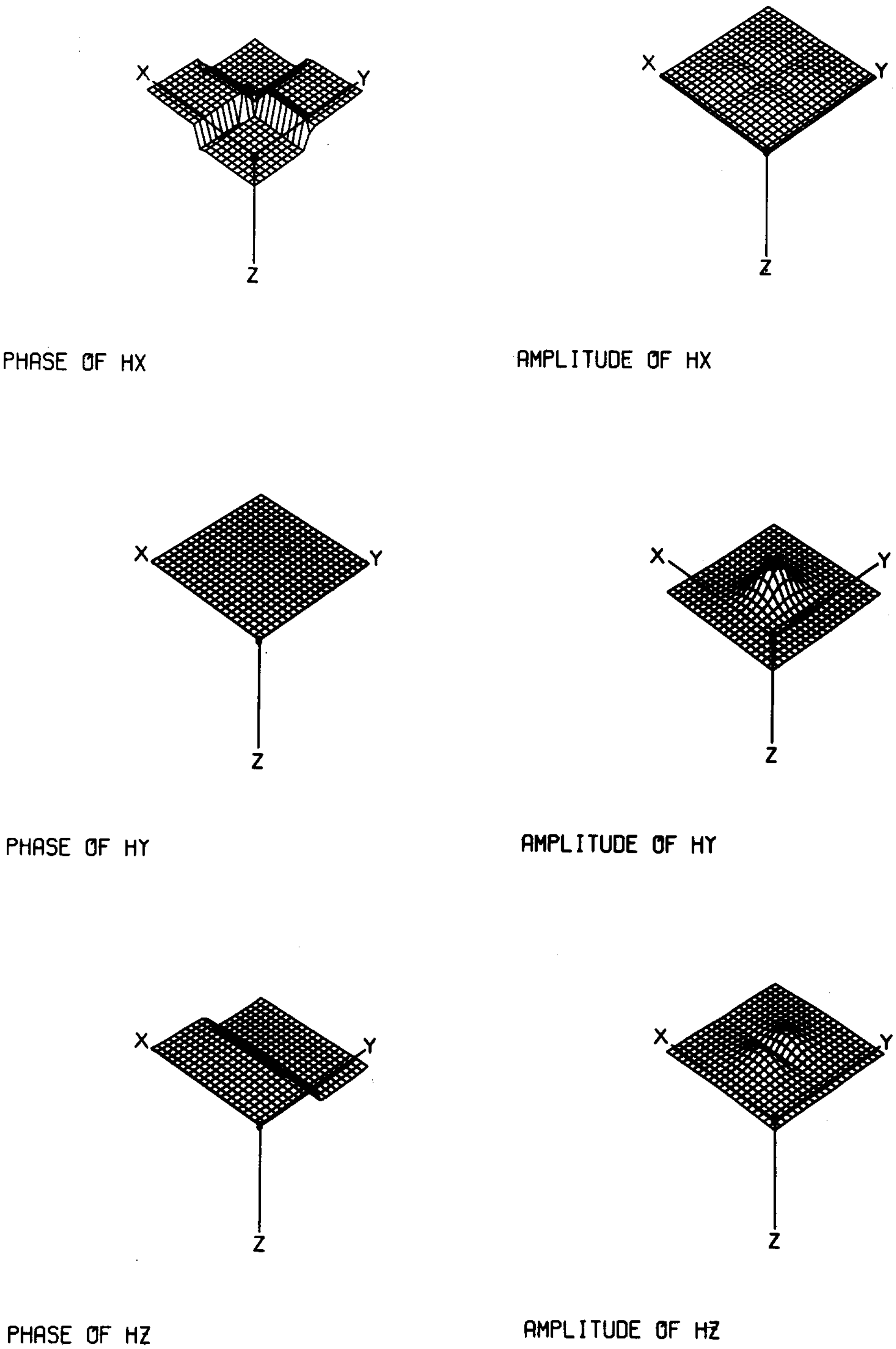

Fig. 1. Magnetic field component amplitudes and phases. Values are calculated over the surface plane of the conductor. The model is of a buried anomaly at the center of the mesh. The anomaly is of conductivity ten times the surrounding medium. The inducing field is such that the electric field is in the x-direction. (After Jones and Pascoe, 1972.)

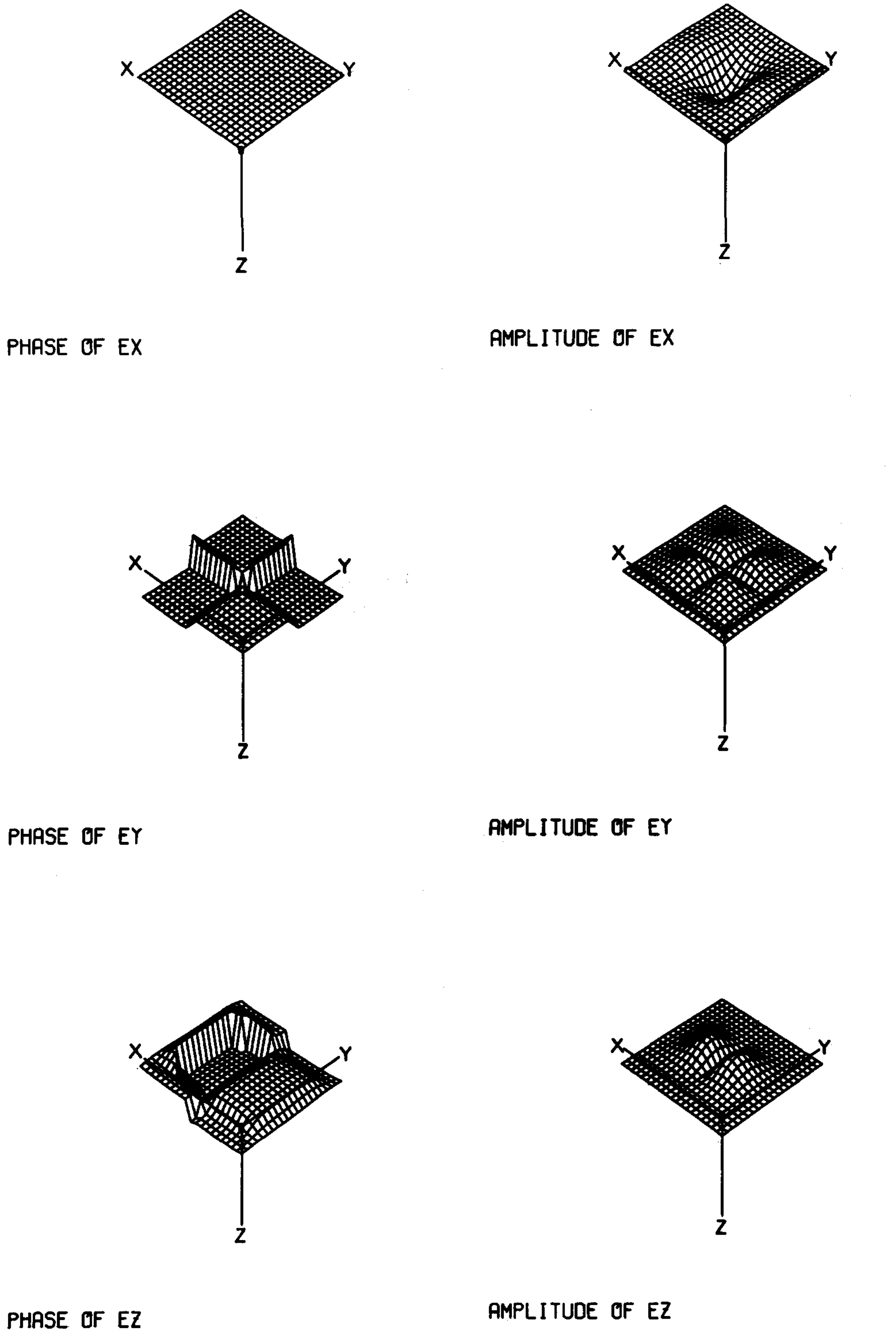

Fig.2. Electric field component amplitudes and phases. Model as in Fig.1. (after Jones and Pascoe, 1972.)

such as the equatorial and auroral areas, we know that the inducing fields are not uniform and are caused by currents of limited spatial extent (Kisabeth and Rostoker, 1971).

Analogue model studies have been made using line current sources (Dosso and Jacobs, 1968) as well as other sources (Thomson, 1972). Hutton (1969) has discussed the case of induction in the earth by the equatorial electrojet and has recently considerably extended this approach including extension to an n-layered earth (Hutton, 1972).

Following the method originally considered by Price (1950, 1962), Hermance and Peltier (1970) have calculated apparent-resistivity curves for induction in a layered earth by a line current, and Peltier and Hermance (1971) have derived expressions to describe the magnetotelluric fields of a Gaussian electroject above a stratified conductor. Schmucker (1971b) has considered models with a non-uniform source over a laterally inhomogeneous earth in which the lateral changes in conductivity are confined to a limited depth range. Hibbs and Jones (1973b) have used the method of Peltier and Hermance (1971) to obtain boundary values for the method of Jones and Price (1970) and Jones and Pascoe (1971) so that perturbations of such fields by embedded inhomogeneities in the E-polarization case may be computed.

Hibbs and Jones (1973c) have recently extended this method to consider a non-symmetric, non-uniform source, as well as aperiodic spatially time-varying sources (Hibbs, 1972; Hibbs and Jones, 1973d).

7. The perturbation of the fields by three-dimensional conductivity inhomogeneities

In the foregoing considerations, only two-dimensional problems have been considered. Many conductivity structures encountered in geophysical studies are essentially two-dimensional in nature and so pursuit of these two-dimensional studies have proven to be, and still are, of great value.

Since 1967, large arrays of magnetic measuring instruments have been used for magnetic deep-sounding studies (Porath et al., 1970; Reitzel et al., 1970; Camfield et al., 1971; Porath and Gough, 1971), and so the magnetic field components at points distributed over considerable areas of the earth's surface may be measured simultaneously. This has led to considerations of the solution of the perturbation problem in three dimensions. Treumann (1970d) has indicated how a solution to the three-dimensional induction problem for a plane earth may be obtained by employing the Green's tensor. Also, Jones and Pascoe (1972), in an extension of their method in two dimensions (Jones and Pascoe, 1971) have presented preliminary results for the three-dimensional problem for buried anomalies where a cubic mesh is used.

In the numerical solution of the three-dimensional problem the amount of information obtained becomes a problem. There are twelve quantities (the six field components and their phases) determined for each grid point, and the number of grid points may be quite large (typically $25 \times 25 \times 25 = 15{,}625$). One method used to display the results is by three-dimensional amplitude and phase diagrams as illustrated in Fig.1 and 2. These diagrams give the magnetic and electric field component amplitudes and phases for a buried anomaly surrounded by a region of uniform conductivity.

Lines and Jones (1973a) have extended the work of Jones and Pascoe (1972) to a grid of variable dimensions and have considered conductive regions of higher contrasts, in particular, island structures. Also, Lines and Jones (1973b) have extended this method so that not only buried and island-type structures may be considered but more general structures that do not necessarily have the same stratification at all boundaries may effectively be dealt with.

In the work by Lines and Jones (1973a,b) three-dimensional amplitude and phase plots as illustrated above are used along with contour plots and profiles of the components to investigate the field behaviour.

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