Physics of the Earth and Planetary Interiors, 7 (1973) 303–312

THEORETICAL MODELS FOR ELECTROMAGNETIC INDUCTION IN THE OCEANS

ATTIA A. ASHOUR
Mathematics Department, Faculty of Science, Cairo University, Cairo (Egypt)

Accepted for publication February 2, 1973

The different methods and techniques employed in the theory of electromagnetic induction in thin sheets are reviewed and the methods for approximation to the solution are indicated. These depend on whether the sheet is closed or finite and on whether the integrated conductivity and/or the frequency of variations is high or low.

Results for induction in finite sheets which are suitable for ocean modelling are given. These include sheets of perfect conductivity and sheets of finite conductivity which is either discontinuous or continuous at the boundary. The dependence of the "coastline effect" for a global ocean on the location of the edge of the continental shelf, the period of variation of the external field and the conductivity of the underlying earth is explained.

1. Introduction

The oceans constitute a surface layer to the earth of relatively high conductivity. This layer, together with the conductivity of the underlying earth modify changes in the earth's magnetic field. Thus the observed change at the earth's surface is the resultant of the primary change and the secondary field of the currents induced in the liquid and solid earth. It is important to separate the primary change (such as the daily variation and the geomagnetic storm) from the secondary. This separation also helps to reveal the surface and other conductivity anomalies near a particular station and provides further information about the conductivity of the underlying earth.

Any idealised theoretical model of this situation is bound to meet with extensive mathematical difficulties. The abrupt change in the conductivity at the coastline and the edge of the continental shelf adds to the complications. Hence such models will only throw light on the order of magnitude of the physical quantities involved.

Several simplifications are made in most of the models considered. The first is that we study the effect of *one* ocean, and not the whole distribution of oceans over the surface of the earth. Second, the ocean is usually represented by a thin conducting layer. The surface, or integrated, conductivity of the layer will depend on the depth of the ocean and hence is not uniform. The integrated conductivity has been assumed infinite in some models, uniform and finite in others and non-uniform and tending to zero at the coastline in the more realistic models.

In most of the models considered so far, the thin sheet was either plane or spherical. The conductivity of the underlying earth is usually represented by a parallel conducting plane or a concentric spherical shell of suitable conductivity at a suitable depth. When only the overall effect of the ocean, but not that of the coastline, is considered the conducting shell representing the ocean is taken closed (i.e., an infinite plane or a complete spherical shell). When both effects are examined, the sheet is taken finite. In the plane models, this finite sheet is taken for mathematical convenience to be either a semi-infinite plane or an infinite strip of uniform width in two-dimensional models and a circular disk in the case of three dimensions. For spherical models, the sheet is usually taken as a spherical cap.

Earlier work on induction in thin sheets includes that of Maxwell (1891) for the uniform plane and spherical sheets and that of Lamb (1887a, b) for the non-uniform circular disk. The most important work since these earlier investigations is that of Price (1949) who obtained the boundary condition at the surface of the sheet in terms of the non-uniform distribution of conductivity and the scalar magnetic potentials of the inducing and induced fields. For application to the ionosphere this condition can be modified to allow for non-isotropic distributions (Ashour and Ferraro, 1962, 1964; Ferris and Price, 1962, 1965). An equivalent boundary condition can also be obtained in terms of the vector potentials of the fields (Ashour, 1971a). Price's condition, or its equivalent, allows the solution for certain non-uniform, finite and closed sheets to be obtained. Other methods of solution have also been suggested. The symmetrical induction problem can be reduced to a Fredholm integral equation (Ashour, 1950). This has been extended recently to cover non-symmetrical problems (Hutson et al., 1972), though they have not actually dealt with any non-symmetrical problem. The induction problem can also be reduced to dual integral or series equations (Ashour, 1965a, b; Weidelt, 1971).

Exact solutions can be obtained for very few distributions of conductivity. These include cases of perfectly conducting finite sheets. For other distributions approximate methods have to be applied. Price (1949) has suggested two general methods to be applied in cases of low and high conductivity. These have been extended recently by Hutson et al. (1972).

Apart from the "thin sheet" model, other model conductors have been considered. These include the important case of the semi-infinite medium with a plane boundary and a vertical discontinuity in the conductivity to resemble the coastline (Weaver, 1971). We shall, however, confine the present paper to the thin sheet models. The following sections of the paper will attempt to explain the methods and techniques of solving the problem of induction in thin sheets and the application to induction in the oceans.

2. The fundamental equations and formulae

2.1. The field equations

The equations for time-varying fields in e.m.u. are:

$$\text{curl}\ \boldsymbol{E} = -\dot{\boldsymbol{B}} \tag{1}$$

$$\text{curl}\ \boldsymbol{H} = 4\pi \boldsymbol{J} + \dot{\boldsymbol{D}} \tag{2}$$

$$\text{div}\ \boldsymbol{B} = 0 \tag{3}$$

We also have:

$$\boldsymbol{B} = \mu \boldsymbol{H}, \ \boldsymbol{J} = k\boldsymbol{E} \tag{4}$$

where $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{B}, \boldsymbol{D}$ and $\boldsymbol{J}$ are the electric field, the magnetic field, the magnetic induction, the displacement vector and the current density, respectively, and μ, k are the permeability and conductivity, respectively.

We assume our conducting sheet to be of infinitesimal thickness d. The medium on both sides is assumed to be non-conducting and the permeability is taken as one everywhere. Further we assume that the time variations of the fields are such that the displacement current can be neglected.

From eq. 3 we thus find that the divergence of the magnetic field vanishes. In the dielectric outside the sheet where $\boldsymbol{J} = 0$, eq. 3 shows that curl $\boldsymbol{H}$ is also zero and hence outside the sheet the magnetic field is derived from a scalar potential satisfying Laplace's equation:

$$\boldsymbol{H} = -\text{grad}\ \Omega, \ \nabla^2 \Omega = 0 \tag{5}$$

From the boundary conditions at the surface of separation of two media (see, for instance, Ferraro, 1954), we find that the tangential components of $\boldsymbol{E}$ and the normal component of $\boldsymbol{H}$ are continuous on crossing the sheet.

The integrated conductivity σ of the sheet and the integrated current density $\boldsymbol{i}$ flowing in its surface are defined as:

$$\sigma = \int_0^d k\,dl, \qquad \boldsymbol{i} = \boldsymbol{i}_s = \int_0^d \boldsymbol{J}\,dl \tag{6}$$

where dl is the element of thickness normal to the surface of the sheet and the subscript s refers to the tangential component. Integration of the equation $\boldsymbol{J} = k\boldsymbol{E}$ through the thickness of the sheet gives:

$$\sigma \boldsymbol{E}_s = \boldsymbol{i}_s \tag{7}$$

From the theory of current sheets (Ferraro, 1954), the relation between the current function Ψ and the magnetic scalar potential of the sheet is given by:

$$\Psi = [\Omega]/4\pi \tag{8}$$

where $[F]$ denotes the change in any function F on crossing the sheet.

2.2. Prices's boundary condition

We assume a conducting sheet as defined earlier.

The sheet may be closed or finite, uniform or non-uniform. We also assume that the sheet is subjected to a time-varying magnetic field of origin external to it. What is required is to determine the field of the currents which is induced in the sheet. The fundamental step towards this end is to obtain the condition to be satisfied by the inducing and induced fields at the surface of the sheet for a given distribution of conductivity. This was done by Price (1949).

Starting with eq. 1 and noting that $\boldsymbol{E}_s$ is a surface function of the sheet, eq. 1 reduces to:

$$\text{curl}\, \boldsymbol{E}_s = -\boldsymbol{n}\, \partial H_n/\delta t = \boldsymbol{n}\, \partial^2\Omega/\partial t\, \partial n \quad (9)$$

where $\partial/\partial n$ denotes differentiation in the direction normal to the sheet and the subscript n refers to the component in that direction and $\boldsymbol{n}$ denotes the unit vector in the same direction. Substitution from eq. 7 in eq. 9 gives after writing $\rho = 1/\sigma$:

$$\text{curl}\,(\rho\, \boldsymbol{i}_s) = \rho\, \text{curl}\, \boldsymbol{i}_s + \text{grad}\, \rho \cdot \boldsymbol{i}_s = \boldsymbol{n}\partial^2\Omega/\partial n \partial t \quad (10)$$

Noting that:

$$\boldsymbol{i}_s = -\boldsymbol{n} \Lambda\ \text{grad}\, \Psi \quad (11)$$

and that from eq. 5 and 8:

$$[\boldsymbol{H}] = -\text{grad}\ [\Omega] \quad (12)$$

we obtain:

$$\rho\ \text{curl}(\boldsymbol{n}\Lambda[\boldsymbol{H}]) + \text{grad}\ \rho\Lambda(\boldsymbol{n}\Lambda[\boldsymbol{H}]) = -4\pi \boldsymbol{n}\partial H_n/\partial t$$

Taking the normal component of both sides yields:

$$\rho\ \text{div}\ [\boldsymbol{H}] + \text{grad}\ \rho \cdot [\boldsymbol{H}] = 4\pi\partial H_n/\partial t \quad (13)$$

Eq. 13 which is satisfied at the surface of the sheet, gives the relation between the varying *total* magnetic fields on the two sides of the sheet irrespective of how they are produced. In our case:

$$\Omega = \Omega^{\mathrm{e}} + \Omega^{\mathrm{i}} \quad (14)$$

where the superscripts e,i refer to the inducing and induced fields respectively. Noting that the inducing field is continuous at the surface of the sheet and that in eq. 9 Ω is replaced by Ω^{i}, eq. 13 reduces to:

$$\rho\nabla^2\Psi + \text{grad}\ \rho \cdot \text{grad}\ \Psi = -\partial^2(\Omega^{\mathrm{e}} + \Omega^{\mathrm{i}})/\partial t\ \partial n \quad (15)$$

at the surface of the sheet.

If the inducing field is given, Ω^{e} will be a given function of the time and the space variables. To find the induced field, it is necessary to determine Ω^{i} which satisfies Laplace's equation, vanishes at a great distance from the sheet and satisfies the boundary condition (eq. 15) at the surface of the sheet.

When the method is applied, Ω^{i} is usually expressed as an infinite series (or integral) of harmonics. When the sheet is closed, the application of eq. 15 usually yields a difference equation for the coefficients of the series. We shall deal later with methods of solution of this difference equation. But when the sheet is finite and forms part of a closed surface, it must be noted that eq. 15 expresses the boundary condition at the surface of the sheet only. On the remaining part of the closed surface, the corresponding condition is that the current density is zero (or the current function is constant).

Price's boundary condition is particularly useful in the application to non-symmetrical problems. The non-symmetry does not add any difficulty. Further, if we have two (or more) concentric spherical sheets (or parallel plane sheets), the boundary condition can be applied at each surface and the solution may be obtained as in the case of one sheet.

2.3. The boundary condition at the surface of the sheet in terms of the vector potential

The boundary condition (eq. 15), due to Price, involves the resistivity of the sheet and its gradient and thus difficulties arise when the resistivity and/or its gradient are discontinuous in the sheet. An equivalent formula which involves only the integrated conductivity of the sheet but not its gradient can be obtained in terms of the vector potentials $\boldsymbol{A}^{\mathrm{e}}$ and $\boldsymbol{A}^{\mathrm{i}}$ (Ashour, 1971a). The details of the derivation will not be given here. This condition* is:

$$[\partial A_s^{\mathrm{i}}/\partial n] = 4\pi\sigma\partial(A_s^{\mathrm{e}} + A_s^{\mathrm{i}})/\partial t \quad (16)$$

where:

$$\boldsymbol{H} = \text{curl}\, \boldsymbol{A},\ \text{div}\, \boldsymbol{A} = 0 \quad (17)$$

and outside the sheet:

$$\nabla^2\boldsymbol{A}^{\mathrm{i}} = 0 \quad (18)$$

The form (eq. 16) of the boundary condition to be satisfied at the surface of the sheet is useful in the

* The particular form of this condition for a uniform plane sheet was given by Smythe (1968).

application to finite sheets with conductivity zero at the edge, i.e., when the conductivity of the closed surface of which the sheet is a part is continuous (but not necessarily its derivatives). This form of the condition is also particularly suitable for application to symmetrical and two-dimensional problems, because in such problems the vector potential has one component only, and is derived from one potential function only.

When non-symmetrical problems are considered, care must be taken that A^i is derived from two independent functions. To illustrate this point, we consider spherical polar coordinates. In these coordinates, the expression for A^i which satisfies eq. 18 with div $A = 0$ is given by Smythe (1968):

$$A^i = \text{curl}\,(\boldsymbol{r}U) + \text{grad}\, V \tag{19}$$

where U and V satisfy Laplace's equation, and $\boldsymbol{r}$ the position vector relative to the origin. The term involving V in eq. 19 does not contribute to the magnetic field and so it is tempting to ignore it. This is justified in the axi-symmetric case because A has one component only in this case. But when both tangential components exist, we cannot neglect V because eq. 16 will be equivalent to two equations, one for each of the components of A^i. In this case, taking V into account affects U and hence indirectly contributes to the magnetic field (Ashour, 1973).

3. Electromagnetic induction in closed sheets

3.1. General method of solution when the scalar or vector potential is expressed as an infinite series of harmonics

If the potentials of the inducing and induced fields are expressed outside the sheet as harmonic solutions of Laplace's equation, then even if the distribution of conductivity assumes a simple form there will not be a one-to-one correspondence between the harmonics of the inducing and induced fields. When the conductivity depends on one surface coordinate only of the sheet, an infinite series of harmonics in the potential of the induced field will generally correspond to each harmonic in that of the inducing field. By substitution in the proper equation (15 or 16) and using the orthogonal properties and recurrence relations of the harmonics, this equation is reduced to a difference equation in the unknown coefficients C_n of the series. In most of the cases this equation is of the form:

$$\left.\begin{aligned} C_{n+1} &= a_n(D)C_n + b_n(D)C_{n-1} \qquad n > 1 \\ C_2 &= a_1(D)C_1 + DH(t) \end{aligned}\right| \tag{20}$$

where D is the operator $\partial/\partial t$, $H(t)$ is a given function of the time depending on the inducing field and a_n and b_n are given functions of D and n. The following method of solution was suggested by Price (1949).

Clearly from eq. 20 any C_n can be expressed in terms of C_1 only. The fact that C_n must vanish as $n \to \infty$ in order that the potential series may converge is used to obtain successive approximations for C_1. Thus it can easily be verified that eq. 20 can be written in the form:

$$C_n = p_n(D)C_1 + Dq_n(D)H(t)\,, \tag{21}$$

where $p_n(D)$ and $q_n(D)$ satisfy the difference equation (eq. 20) and have given initial values, namely:

$$p_1 = 1, \quad p_2 = a_1; \quad q_1 = 0, \quad q_2 = 1 \tag{22}$$

In fact $p_n(D)$ and $q_n(D)$ are polynomials of degree n and $n-1$ in D. Hence, since $C_n \to 0$ as $n \to \infty$:

$$C_1 = -\lim_{n\to\infty} \frac{Dq_n(D)}{p_n(D)} H(t) \tag{23}$$

Successive approximations C_{1s} to C_1 and C_{ns} to C_n are obtained by taking $C_s = 0$. These are given by:

$$\left.\begin{aligned} C_{1s} &= -D\frac{q_s(D)}{p_s(D)} H(t)\,, \qquad s = 1, 2, \ldots \\ C_{n,s} &= p_n(D)C_{1s} + Dq_n(D)\,H(t), \\ &\qquad\qquad n = 1, 2, \ldots, s-1 \end{aligned}\right| \tag{24}$$

When the fields vary harmonically, the operator D can be taken in the steady state as equal to a pure imaginary constant and the process of approximation is continued till the coefficients and the series giving the components of the fields as calculated from two consecutive steps differ negligibly. On the other hand, if we require to calculate the effects of a sudden rise in the inducing field, i.e., if $H(t)$ is given by:

$$H(t) = 0 \qquad t < 0$$
$$= H_0 \qquad t \geq 0 \qquad (25)$$

we need to apply the Heavyside operational rule, namely:

$$\frac{D}{D+\alpha} H(t) = H_0 \, e^{-\alpha t} \qquad (26)$$

If the operator $q_s(D)/p_s(D)$ can be expressed as a sum of partial fractions and eq. 26 is applied C_{1s} and C_{ns} will be expressed in the forms:

$$C_{1s} = H_0 \sum_{r=1}^{s} A_{sr} \, e^{-\alpha_{sr} t} \qquad (27)$$

$$C_{ns} = H_0 \sum_{r=1}^{s} A_{sr} \, p_n(-\alpha_{sr}) \, e^{-\alpha_{sr} t}$$
$$n = 1, 2, \ldots s-1 \qquad (28)$$

where $-\alpha_{sr}(r = 1, 2, \ldots, s)$ are the zeros of $p_s(D)$ and:

$$A_{sr} = q_n(-\alpha_{sr})/p_n(-\alpha_{sr})$$

It is to be noted here that in this method it is implied that the time constants of the sheet are discrete (i.e., the α_{sr} are all different, real and positive). If this is the case, one usually obtains a good approximation from a few steps. Otherwise, i.e., if the time constants for the particular conductivity distribution considered are not discrete, the method will not converge.

The above procedure has been successfully applied by Ashour and Price (1948) for induction in a spherical shell with $\sigma = \sigma_0/(1 + \epsilon \cos\theta)$ by a non-symmetrical field, by Price (1949) for induction in a plane sheet for which σ varies harmonically with x and in several other problems of closed sheets of continuous conductivity (Ashour and Ferraro, 1964; Ashour, 1969).

An alternative method to solve the difference equation (eq. 20) is to regard it as a system of infinite linear equations (Ashour, 1971 a). The method of approximating to the solution in this case is to neglect C_n for $n > s$ and consider the first s equations only *. The process is continued till it is found that the series giving the components of the field as calculated from two consecutive steps s and $s+1$ do not differ appreciably. The advantage of this procedure is that all the coefficients are obtained in one step and so any error in C_{1s} does not pile up when calculating the following coefficients as may happen in the first method of solution.

* See for instance Kantorovitch and Krylov (1964).

3.2. Iterative methods of solution

Price (1949) suggested two iterative methods of solution to the problem of induction in thin sheets for the two cases $\omega\sigma$ small or large ($\omega = |D|$). These methods are explained in Professor Price's review in the present issue. Here we only give the relations connecting two consecutive steps (s and $s+1$) in the iteration process in each of the two cases. When $\omega\sigma$ is small, we have using Price's boundary condition:

$$\nabla^2\Psi_{s+1} + \sigma \operatorname{grad} \sigma^{-1} \cdot \operatorname{grad} \Psi_{s+1} = -\sigma\omega(H^e + H^i_s) \qquad (29)$$

and when $\sigma\omega$ is large, we have:

$$H^i_{s+1} = -H^e + (\omega\sigma)^{-1}(\nabla^2\Psi_s + \sigma \operatorname{grad} \sigma^{-1} \cdot \operatorname{grad} \Psi_s) \qquad (30)$$

where the subscript n has been omitted.

The two procedures have been applied by Price to a special case for which an exact solution is known, namely that of the uniform infinite plane sheet and in both cases the process of approximations converged to the solution.

Most of the other iterative solutions which have been suggested including those for finite sheets, fall within the above two methods. Further discussion of iterative methods of solutions will be given later in connection with finite sheets.

Bullard and Parker (1970) modified eq. 29 slightly by expressing H^i_s on the L.H.S. in terms of the current function Ψ_s. They also extended eq. 29 to cover the case when there is a perfectly conducting concentric sphere in addition to the spherical finite sheet.

3.3. Surface integral formulae

Hobbs and Price (1970) derived surface integral formulae expressing any one of certain field quantities, namely current functions and normal components of magnetic fields, in terms of any one other for currents flowing in concentric spherical surfaces. These formulae are useful in obtaining the vertical magnetic field, which is usually represented by series which are either divergent or slowly convergent at lines of discontinuity of the conductivity, in terms of the current function whose series is usually quickly convergent. Hobbs (1971) successfully applied these integral formulae to electromagnetic induction in a hemispherical shell representing the Pacific Ocean surrounding a concentric perfectly conducting sphere.

4. Finite sheets

4.1. Perfectly conducting sheets

When the sheet is perfectly conducting, the boundary condition at its surface reduces to the vanishing of the total normal magnetic field. Hence the induction problem in this case is the same as that of the irrotational flow of a perfect inviscid fluid across the sheet and is therefore relatively simpler than when the conductivity is finite. Exact solutions can be obtained in special cases.

In the case of the infinite strip of uniform width and the circular disk subjected to a normal uniform inducing field, simple solutions can be obtained using elliptic and oblate spheroidal coordinates because these two sheets are natural boundaries in the two systems respectively (Smythe, 1968; Lamb, 1945). For more complicated inducing fields, the exact solutions are obtained by reducing the problem to a system of dual integral equations (Ashour, 1965a).

When the sheet is in the form of a spherical cap, the solution for symmetrical inducing fields can be obtained from that for the disk or independently (Collins, 1961; Ashour, 1965a, b; Doss and Ashour, 1971). The non-symmetric case has also been solved by reducing the problem to a system of integral equations (Ashour, 1965b, c; Doha, 1972).

The results for perfectly conducting sheets can be summarised as follows:

(1) The vertical component of the total field is zero in the sheet and is infinite just outside its boundary.

(2) The tangential component of the total field normal to the boundary of the sheet tends to ∞ at the boundary of the sheet and is negligible outside it.

(3) The tangential component parallel to the boundary is finite and continuous at the boundary.

(4) The current density is infinite at the boundary but the total current in the sheet is finite.

The infinities appearing in the components and the current density at the boundary are all of order $x^{-\frac{1}{2}}$ as $x \to 0$.

4.2. Exact solutions for certain finitely conducting sheets

4.2.1. Lamb's solution for the circular disk

Lamb (1887b) considered the principal time constant of a circular disk and restricted himself to axisymmetric currents decaying freely in the disk. He showed from simple physical principles that if the surface conductivity at distance ρ from the centre is given by:

$$\sigma(\rho) = \sigma_0(1 - \rho^2/a^2)^{n-\frac{1}{2}}/F(-n, \tfrac{3}{2}, 2, \rho^2/a^2) \qquad (31)$$

the current density will be given by:

$$i(\rho) = \frac{\pi}{a}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\cdot\frac{\rho}{a}\left[1 - \frac{\rho^2}{a^2}\right]^{n-\frac{1}{2}} \qquad (32)$$

and the principal time constant of the sheet by:

$$\tau = \Gamma(n+\tfrac{1}{2})\, a\, \pi^{3/2}\, \sigma_0/2\,\Gamma(n+1) \qquad (33)$$

where $n + \frac{1}{2} > 0$ and F is the hypergeometric function.

Two cases of special importance have been considered by Lamb, namely $n = \frac{1}{2}$ and $n = 1$.

When $n = \frac{1}{2}$ the conductivity is finite everywhere in the sheet and does not vanish at the edge. The corresponding current density is clearly finite at the edge.

When $n = 1$, the conductivity is given by:

$$\sigma(\rho) = 4\sigma_0(a^2 - \rho^2)/(4a^2 - 3\rho^2) \qquad (34)$$

This distribution is almost constant all over the surface of the disk and falls sharply to zero at the edge, i.e., it is continuous at the edge. This distribution is very suitable for ocean modelling. The current density is zero at the edge and it can be proved (Ashour, 1971a) that the field induced in such a sheet is also finite and continuous there.

4.2.2. Weidelt's solution for two thin half sheets

Weidelt (1971) considered electromagnetic induction in two adjacent half-sheets with different uniform conductivities. His system of conductors also included a perfectly conducting parallel sheet. The inducing field is such as to render the problem two-dimensional.

The problem was reduced to two dual integral equations which could be solved using a method of contour integration due to Clemmow (1951, 1953). The results obtained by Weidelt show that as long as the conductivities of the two half-sheets are finite and unequal, the current density and the tangential magnetic field are both finite and discontinuous at the line of discontinuity. The normal component of the field has a logarithmic infinity at the line of discontinuity and on both sides.

A similar result was also rigorously obtained by Parker (1968) who considered the two-dimensional problem of induction in an infinite strip by a parallel symmetrical current line.

4.3. The reduction of the problem to a Fredholm integral equation and allied methods

When the sheet is in the form of a surface of revolution and both the conductivity and inducing field have axial symmetry, the problem of induction in the sheet can be reduced to that of a Fredholm integral equation (Ashour, 1950). This is done by regarding the sheet as composed of an infinite number of coaxial annular circuits. Thus if the equation of the surface of the sheet in cylindrical coordinates is:

$$z = f(\rho) \tag{35}$$

and its integrated conductivity is $\sigma(\rho)$, the integral equation for the current density $i(\rho)$ is:

$$i(\rho) = -\rho DH(\rho, t)\,\sigma - \frac{\sigma D}{2\pi\rho}\int_{\rho_1}^{\rho_2} i(\rho')M(\rho, \rho')\,(1 + f'^2(\rho'))^{\frac{1}{2}}\,d\rho' \tag{36}$$

where ρ_1 and ρ_2 are the radii of the boundaries of the sheet, $H(\rho, t)$ is the flux across the circuit of radius ρ due to the inducing field, $M(\rho, \rho')$ is the coefficient of mutual induction between the two circuits of radii ρ, ρ' and $D = \partial/\partial t$ as before.

The above equation can easily be transformed to another Fredholm integral equation with symmetrical kernel (Whittaker and Watson, 1920) of the form:

$$g(\rho) = \lambda F(\rho) + \lambda \int_a^b g(\rho')K(\rho, \rho')\,d\rho' \tag{37}$$

where λ varies with the conductivity and the frequency. The kernel of either of these two equations has an infinity of order $\log|\rho - \rho'|$ when $\rho = \rho'$. This does not however raise difficulties when applying Fredholm's solution, because it was proved by Hilbert (1924) *, that these infinities can be replaced by zeroes without altering the solution of the equation provided that a positive number $\beta < \frac{1}{2}$ can be found such that:

$$\lim_{\rho \to \rho'} (\rho - \rho')^\beta K(\rho, \rho') = 0 \tag{38}$$

Condition (38) is clearly satisfied in our case.

When λ is small, i.e., for low conductivity and slowly varying fields the method of continued substitution, namely:

$$g_{n+1}(\rho) = \lambda F(\rho) + \lambda \int_a^b g_n(\rho')K(\rho, \rho')\,d\rho' \tag{39}$$

can be used to obtain the approximate solution of the equation. In fact Hille and Tamarkin (1930) have proved that when $K(\rho, \rho')$ is symmetric and has discontinuities when $\rho = \rho'$, eq. 37 admits a convergent solution by the method of continued substitution, provided that: (1) $F(\rho)$ is integrable; and (2) $1/(b-a) \int_a^b K(\rho, \rho')\,d\rho < |\lambda^{-1}|$.

Ashour (1950) considered the special case of a uniform circular disk subjected to a uniform inducing field normal to it and calculated the amplitude and phase of the current density by using the method of continued substitution and also Fredholm solution. He also estimated the principal time constant of the sheet. For an ocean of global dimensions this is about 4 hours.

Two-dimensional problems can also be reduced to the solution of a Fredholm integral equation by regarding the sheet as composed of an infinite number of thin strips. Roden (1964) considered induction in a uniformly conducting strip with a perfectly conducting plane underneath it. He reduced the problem to a Fredholm integral equation and his solution is similar to that of Ashour (1950) for the circular disk.

It should be noted here that the method of continued substitution used by Ashour (1950) and by Roden (1964) is in fact identical with the method suggested by Price (1949) for low conductivity and slowly varying fields.

* See also Bôcher (1914).

When $\sigma\omega$ is large, the method suggested by Price is met in the case of finite sheets by the difficulty that the first approximation, which corresponds to the case of infinite conductivity, involves infinities in the current density and induced field at the boundary of the sheet.

Doss and Ashour (1971) considered induction in a finitely and highly conducting hemispherical shell with the aim of finding the range of frequencies for which an ocean of global dimensions acts as if of infinite conductivity except near the coastline. The problem was reduced to two dual Bessel integral equations. To avoid the above-mentioned difficulty, a special inducing field was assumed for which no infinities appear at the boundary. The second approximation was obtained for a value $4\pi a\sigma\omega = 100$. This corresponds to a period of 2.4 hours. The results obtained differed negligibly, except near the coastline, from those for an ocean of infinite conductivity. Hence it was concluded that for the overall effect of the ocean, it can be taken as perfectly conducting for periods less than 2.4 hours*.

The method of the Fredholm integral equation has been recently extended, though not applied, to cover non-symmetrical problems by Hutson et al. (1972). The equation they obtain for a general sheet involves the scalar electric potential in addition to the current density. For the symmetrical case, this equation reduces exactly to that derived by Ashour. The authors considered the integral equation for a symmetrical problem of induction in a hemispherical shell and gave a proof, using theorems in Functional Analysis, that the current density is finite at the boundary** of the sheet. They also gave a modification to the method of continued substitution to cover the case of high conductivity, thus extending Price's first method to apply to this case. This modification is simply subtracting αg_{n+1} and αg_n from the left- and right-hand sides of eq. 37. It was stated that α, an arbitrary constant, could always be chosen to give a convergent series of approximations provided that neither the conductivity nor the frequency is infinite. No physical significance was however given for α. Also no way was indicated of how to choose an optimum α for a particular problem. The method was illustrated by a numerical case for which $4\pi a\sigma\omega = 16$. This, for a global ocean, corresponds to variations of period about 15 hours, and thus could have been adequately dealt with by the ordinary continued substitution method ($\alpha = 0$). This does not, however, lessen the potentialities of the method and it is hoped that it will be applied to problems which could not be solved by previous methods. It is also hoped that the method of calculating the field components at the boundary, and not only the current density, will be clarified in future publications.

* Due to an arithmetic error, it was mentioned in the paper that the value 10^{-2} for $1/4\pi a\sigma\omega$ corresponds to a 24 hours period. The results obtained are, as mentioned here, for a 2.4 hours period.

** Doss and Ashour (1971) were criticised by the authors on the grounds that they claim that the current density is infinite at the boundary. In fact, this was not claimed. The infinity which appears is the result of the second approximation and cannot be taken as the rigorous solution at the boundary. Further, reference was given by Doss and Ashour to Parker (1968) and Weidelt (1971) who proved that the current density is finite at the boundary.

4.4. Results for finitely conducting sheets with discontinuous conductivity at the edge

The results for sheets with finite but discontinuous conductivity at the edge can be summarised as follows:

(1) The current density and tangential components of the induced field are finite at the boundary of the sheet and have discontinuities on crossing it.

(2) The vertical component of the inducing field has a logarithmic infinity at the boundary, this component is reversed at the boundary.

4.5. Sheets with conductivity decreasing to zero at the edge

A more realistic model for representing the ocean is a finite sheet for which the conductivity decreases to zero at the edge and thus is continuous there. Ashour (1971a) considered such sheets in the forms of a circular disk, an infinite strip of uniform width and a hemispherical shell. The conductivity for the disk was that assumed by Lamb (1887b), and given by eq. 34.

It can easily be proved that for such sheets the current density is zero at the edge. Using spheroidal harmonics (in the case of the disk), applying the boundary condition (eq. 16) in terms of the vector potential and starting by an expansion for the current density which vanishes at the edge, it was proved that the magnetic field is finite and continuous at the boundary. The problem was reduced to a difference equation of the form of eq. 20. For the geomagnetic application an ocean of global dimensions and a uniform (or dipole) inducing field with time variations of 1–24 hours or sudden changes were considered. The results which were obtained from a satisfactory number of terms of the series show that:

(1) The induced vertical component is enhanced manyfold at the coastline inland and in sea. It is not reversed at the coastline (as in the case of a discontinuity in the conductivity there), but in the region around the edge of the continental shelf. The observed total vertical component is not reversed.

(2) The horizontal component is enhanced is sea only just off the coast.

In a following paper, Ashour (1971b) considered a model which consists of a hemispherical ocean covering a pertectly conducting concentric sphere. The distribution of conductivity in the outer conductor was given by:

$$\sigma(\theta) = \sigma_0(1 + \alpha) \cos\theta (1 + \alpha \cos^2 \theta)^{-1} \quad (40)$$

where α is a constant. Thus $\sigma(\theta)$ vanishes at the edge and by changing α, the location of the edge of the continental shelf could be changed and its effect studied. Also, the effect of the depth of the effective conducting mantle could be examined by changing the radius of the inner conductor. The problem was reduced to an infinite system of linear equations and the results for a global ocean indicate, in addition to the results of Ashour (1971a), that:

(1) The position of the edge of the continental shelf is a very important factor in the shielding of geomagnetic variations by a large ocean. For periods of more than one hour, the enhancement in the observed total field in sea and land is minimal unless this position is sufficiently near the coastline.

(2) The enhancement of the components of the total field is more and occurs nearer to the coastline for the shorter periods of variation. For periods of one hour or less the effects are almost the same as for a perfectly conducting ocean except that the field is finite at the coastline.

(3) The coupling between the earth and ocean reduces the screening effects of the ocean by a factor ranging between 20 and 50% *. Hence the enhancement of the total field for periods up to 24 hours, though reduced when both ocean and earth are taken into account, should be observable for stations in the location of the coastline provided the shelving of the ocean does not occur far from it.

5. Conclucions

For improving our analytical techniques in the theory of induction in thin sheets we need to know more about integral equations, dual integral and series equations, infinite system of linear equations and their numerical solution.

The utmost that the theory of thin sheets can offer for electromagnetic induction in the oceans is to find the currents and the field induced by the different geomagnetic field variations in a spherical shell whose integrated conductivity at any point varies with the depth of the liquid earth at that point, and in an inner conductor representing the conductivity of the solid earth to the best of our knowledge. This is an ambitious and difficult task, but even if it is accomplished, we should not forget the limitations of the thin-sheet model (Schmucker, 1970). Model experiments may have more to offer in this respect.

References

Ashour, A.A., 1950. *Q.J. Mech. Appl. Math.*, 3:119.
Ashour, A.A., 1965a, *Q.J. Mech. Appl. Math.*, 18:73.
Ashour, A.A., 1965b. *Proc. Lond. Math. Soc.*, 14 (3):557.
Ashour, A.A., 1965c. *Geophys. J.*, 10:147.
Ashour, A.A., 1969. *Z. Geophys.*, 35:269.
Ashour, A.A., 1971a. *Geophys. J.*, 22:417.
Ashour, A.A., 1971b. *Geophys. J.*, 25:447.
Ashour, A.A., 1973. *Geophys. J.*, in press.
Ashour, A.A. and Ferraro, V.C.A., 1962. *Nature*, 196:260.
Ashour, A.A. and Ferraro, V.C.A., 1964. *J. Atmos. Terr. Phys.*, 26:509.
Ashour, A.A. and Price, A.T., 1948. *Proc. R. Soc. Lond., Ser. A*, 195:198.
Bôcher, M., 1914. *Cam. Tracts Math.*, 10.
Bullard, E.C. and Parker, R.L., 1970. Electromagnetic induction in the oceans. In: A.E. Maxwell (Editor), *The Sea, Vol. 4, Ch. 18.*, Wiley, New York, N.Y.

* This is in agreement with Rikitake (1961).

Clemmow, P.C., 1951. *Proc. R. Soc. Lond., Ser. A,* 205:286.

Clemmow, P.C., 1953. *Philos. Trans. R. Soc. Lond.,* 246:1.

Collins, W.D., 1961. *Proc. Cam. Philos. Soc.,* 57:367.

Doha, E.H., 1972. Thesis, Faculty of Science, Cairo Univ.

Doss, S.S. and Ashour, A.A., 1971. *Geophys. J.,* 22:385.

Ferraro, V.C.A., 1954. *Electromagnetic Theory.* Athlone Press.

Ferris, G.A. and Price, A.T., 1962. *Nature,* 196:258.

Ferris, G.A. and Price, A.T., 1965. *Geophys. J.,* 9:285.

Hilbert, D., 1924. *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen.* Herf, Leipzig.

Hille, E. and Tamakin, J., 1930. *Ann. Math.,* 31:479.

Hobbs, B.A., 1971. *Geophys. J.,* 25:481.

Hobbs, B.A. and Price, A.T., 1970. *Geophys. J.,* 20:49.

Hutson, V.C.L., Kendall, P.C. and Malin, S.R.C., 1972. *Geophys. J.,* 28:489.

Kantorovich, L.V. and Krylov, V.I., 1964. *Approximate Methods of Higher Analysis.* Interscience, New York, N.Y.

Lamb, H., 1887a. *Philos. Trans. R. Soc. Lond.,* 178:31.

Lamb, H., 1887b. *Proc. R. Soc. Lond., Ser. A,* 42:289.

Lamb, H., 1945. *Hydrodynamics.* Cambridge.

Maxwell, J., 1891. *Electricity and Magnetism.* Oxford, 3rd edition.

Parker, R.L., 1968. *Geophys. J.,* 14:487.

Price, A.T., 1949. *Q.J. Mech. Appl. Math.,* 2:283.

Rikitake, T., 1961. *Geophys. J.,* 5:1.

Roden, R.B., 1964. *Geophys. J.,* 8:375.

Schmucker, U., 1970. *Bull. Scripps Inst. Oceanogr., Univ. Calif.,* 13.

Smythe, W.R., 1968. *Static and Dynamic Electricity.* McGraw-Hill, New York, N.Y.

Weaver, J.T., 1971. *Geophys. J.,* 22:100.

Weidelt, P., 1971. *Z. Geophys.,* 37:649.

Whittaker, E.T. and Watson, G.N., 1920. *Modern Analysis.* Cambridge.