GEOPHYSICS, VOL. 36, NO. 1 (FEBRUARY 1971), P. 156 165, 4 FIGS.

INTERPRETATION OF INDUCTION ANOMALIES ABOVE NONUNIFORM SURFACE LAYERS†

ULRICH SCHMUCKER*

Using convolution integrals to account for the inductive coupling between surface layer and substratum, we derive the conductance of a nonuniform surface layer above a layered substratum by direct inversion from the system's anomalous inductive response to natural geomagnetic variations. The conductivity in the substratum and the uniform conductance of the surface layer at some distance from the nonuniformity (two-dimensional) must be known. The convolution method is also applied to the reverse problem: finding the anomalous inductive response for a given two-dimensional nonuniformity. Our calculations are based on Price's thin-layer approximation.

INTRODUCTION

We are concerned here with the inductive response of electrically conducting matter below the earth's surface to natural fluctuations in the geomagnetic field originating from sources in the ionosphere and magnetosphere or from artificial sources used in geophysical prospecting. In anomalous zones, where the otherwise stratified flow of induced eddy currents is perturbed by lateral changes of the conductivity, we observe at the earth's surface a local perturbation of the inductive response, i.e., an *induction anomaly* of geomagnetic and geoelectric time variations.

The following contribution presents a new method for the interpretation of two-dimensional induction anomalies of superficial origin ("surface anomalies"). Such anomalies arise, for instance, near coastlines and on islands from the conductivity contrast between seawater and rock formations. Similar anomalies are also found inland where they reflect the varying thickness and conductivity of unfolded sediments above a highly resistive crystalline basement. It will be assumed here that the deep conductivity structure beneath the nonuniform surface cover is without lateral gradients and that the conductivity as a function of depth in the underlying crust and upper mantle is known. The downward diffusion of the surface perturbation into the substratum, therefore, is a "normal" induction problem. By solving the resulting diffusion equation for the substratum with standard methods, we obtain a boundary condition for the anomalous variation field at the inner surface of the nonuniform cover, which takes into account the inductive coupling between the cover and the underlying substratum. The boundary condition for the outer surface of the cover follows from the fact that the induction anomaly is of purely internal origin and that the magnetic variation fields to be considered here may be regarded as irrotational above the earth's surface, obeying Laplace's equation.

The "anomalous" induction problem in the nonuniform surface cover itself will be treated in accordance with Price's (1949) original work on this subject. Thus, the lateral nonuniformity will be contained in a variable total conductivity

$$\tau = \int_0^d \sigma(z)dz, \tag{1}$$

where d denotes the thickness of the surface cover and $\sigma(z)$ is the conductivity as function of depth. The vertical magnetic and the tangential electric components of the variation field are taken to be the same at corresponding points just above and

† Manuscript received by the Editor May 11, 1970; revised manuscript received July 21, 1970.

* University of Gottingen, Gottingen, Germany.

beneath the surface cover, which is regarded as being infinitely thin. This approximation can be used when, for the frequency f, the mean skin depth p of the surface material is several times larger than its thickness d: $p=1/\sqrt{\pi\mu_0\sigma f}=30.2/\sqrt{\sigma f}$ in km, when f is measured in cycles per hour (cph) and the conductivity σ in $(\text{ohm-m})^{-1}$. Since the conductivity of unconsolidated sediments can reach 1 $(\text{ohm-m})^{-1}$ and the conductivity of seawater is roughly 4 $(\text{ohm-m})^{-1}$, the ultimate permissible frequencies for this approximation are about 8 cph for inland anomalies and 4 cph for those near coastlines, where the depth of the overlying material is 6 km. The depth of penetration of the variation field into the underlying crust and mantle should be likewise large in comparison with d. We can expect that this second condition for Price's approximation is everywhere satisfied in view of the high resistivity of the crust down to considerable depth.

A further example, this time from prospecting geophysics, is the case of an overburden layer over a more resistive bedrock. The approximations and hence the proposed interpretation are valid, for example, at a frequency of 1000 hz for a 50 m layer of overburden of conductivity 10^{-2} $(\text{ohm-m})^{-1}$ lying on an infinite half-space (basement) of conductivity 10^{-4} $(\text{ohm-m})^{-1}$. All equations are written in rational mks-units with μ_0 as the free-space magnetic permeability.

BASIC EQUATIONS

Consider right-handed Cartesian coordinates with z down (Figure 1). An infinitely thin sheet with variable total conductivity $\tau(y)$, representing oceans or geological formations above the crystalline basement, occupies the (x, y) plane between

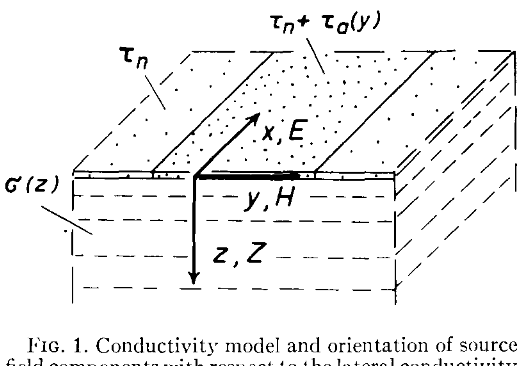

FIG. 1. Conductivity model and orientation of source field components with respect to the lateral conductivity gradient; τ is the total conductivity of a thin surface sheet.

a nonconducting upper half-space (air) and a layered conducting lower half-space (crust, mantle). The nonuniformity of the surface sheet shall be restricted to a certain limited range in y,

$$\tau(y) = \tau_n + \tau_a(y), \tag{2}$$

with $\tau_n=\tau(-\infty)$ as the uniform normal part of the total conductivity. If the nonuniformity lies between two uniform but different sections of the sheet, the anomalous part $\tau_a(y)$ approaches a constant, $\tau(+\infty)-\tau(-\infty)$, as $y\to+\infty$. The normal part τ_n is assumed to be known and the anomalous part $\tau_a(y)$ is to be found from the perturbed inductive response of sheet and substratum to a slowly oscillating electromagnetic field which has its primary source in the upper half-space.

Let the source field be two-dimensional and with the E-polarization perpendicular to the lateral gradient of τ, i.e., the electric vector and the flow of induced currents are in the x direction, while the magnetic vector is confined to the vertical (y, z) plane as shown in Figure 1. The derivatives of the field components and of the internal conductivity distribution with respect to x are zero.

The scalar electric field E in the x direction and the vertical magnetic field Z in the z direction pass as functions of time t without change from the upper into the lower half-space. The horizontal magnetic component H in the y direction, however, changes discontinuously from $H^+=H(t, y, -0)$ just above the sheet $(z=-0)$ to $H^-=H(t, y, +0)$ just below the sheet $(z=+0)$, because of the induced sheet current in the (x, y)-plane. Let j denote the density of the sheet current per unit breadth in the x direction. Then the field equations to be solved for the (x, y) plane are

$$H^+ - H^- = j, \tag{3a}$$

$$\partial E/\partial y = \mu_0\dot{Z}, \tag{3b}$$

with

$$j = \int_0^d E(z)\sigma(z)dz,$$

as the thin-sheet version of the constitutive equation. $\dot{Z}$ denotes the time derivative of Z. Since E has been assumed constant over the thin layer, it may be taken out of the integral and, using equation (1), we may write j as

$$j = \tau E. \tag{4}$$

From the fact that H is divergenceless, we have

$$\partial H/\partial y + \partial Z/\partial z = 0 \tag{5}$$

throughout the upper and lower half-spaces. For the lower conducting half-space, Maxwell's field equations are combined (with the neglect of displacement currents) into a two-dimensional diffusion equation for the electric and magnetic variation field:

$$\nabla^2 F = \mu_0 \sigma \partial F/\partial t, \tag{6}$$

where F can be E, Z, or H and $\sigma = \sigma(z)$ within a layered substratum (Appendix A).

These basic equations will be solved separately for the *normal* and *anomalous* parts of the variation field. Here the normal parts refer to the inductive response of the sheet and substratum without the nonuniformity $\tau_a(y)$. They are denoted with the subscript "n", e.g., E_n. The superimposed anomalous parts, which account for the perturbation of the inductive response due to $\tau_a(y)$, carry the subscript "a", e.g., $E_a = E - E_n$.

The separation into normal and anomalous parts, when applied to equation (4), leads to

$$j_n = \tau_n E_n, \tag{7a}$$

$$j_a = \tau_a(E_n + E_a) + \tau_n E_a, \tag{7b}$$

as the normal and anomalous part of the induced sheet current density. It will be seen that on the basis of a given deep conductivity structure a second equation connecting j_a and E_a can be formulated, which when combined with equation (7b), allows the elimination of j_a, yielding $\tau_a(y)$ in terms of E_a, E_n, and τ_n. In this way the unknown perturbation τ_a can be found, for a given normal value τ_n, from surface observations of an induction anomaly in the electric field.

We note at this point that the anomalous field components E_a, Z_a, and H_a^+, when considered as functions of y in the upper $z = -0$ plane, are not truly independent and that their interdependence is not connected with the internal conductivity distribution. The second field equation (3b) implies that we can obtain the time derivative of Z_a (or Z_n) from $E_a(E_n)$ by a differentiation with respect to y and conversely can find $E_a(E_n)$ from $\dot{Z}_a(\dot{Z}_n)$ by integration [cf. also equation (18b)]. Furthermore, Z_a can be derived from H_a^+ as function of y or vice versa, because the magnetic induction anomaly must be of purely internal origin [cf. equation (15)]. Hence, the intended inversion to determine $\tau_a(y)$ from surface observations can be carried out by starting with a profile of either one of the anomalous and normal field components.

Let us add the field equations (3) for a field which varies sinusoidally in time and in the y direction. Setting $F(t, y, 0) \sim \hat{F}(\omega, k) \cdot \exp(i[\omega t + ky])$ for $F = E$, Z, or H in the (x, y)-plane, we obtain

$$\hat{H}_n^+ - \hat{H}_n^- = \hat{j}_n, \qquad \hat{H}_a^+ - \hat{H}_a^- = \hat{j}_a, \tag{8a}$$

$$k\hat{E}_n = \omega\mu_0 \hat{Z}_n, \quad \text{and} \qquad k\hat{E}_a = \omega\mu_0 \hat{Z}_a \tag{8b}$$

as field equations for the Fourier spectra of the anomalous and normal parts in the frequency-wavenumber domain (see below). These spectra follow from the original time-distance functions by the successive transformations

$$\tilde{F}(\omega, y) = \int_{-\infty}^{+\infty} F(t, y, 0) \exp(-i\omega t) dt$$

and

$$\hat{F}(\omega, k) = \int_{-\infty}^{+\infty} \tilde{F}(\omega, y) \exp(-iky) dy;$$

$\omega = 2\pi f$ is the angular frequency and k, a spatial wavenumber. Since the field to be considered is slowly oscillating, k is *not* the wavenumber of a truly propagating wave but simply accounts for the surface modulation of a quasi-stationary transient field. The notations $\tilde{F}$ and $\hat{F}$ will be used throughout this presentation to identify spectral functions in the frequency-distance and frequency-wavenumber domains, respectively.

BOUNDARY CONDITIONS

The field equations of the previous section are supplemented by two boundary conditions which apply to the anomalous magnetic field just above and below the nonuniform sheet in the (x, y)-plane. They will be formulated for the Fourier spectra of Z_a and H_a in the (ω, k) domain and subsequently, after an inverse Fourier transformation, in the (ω, y)-domain.

The inductive response of the conducting lower half-space to a nonuniform transient surface field can be expressed by a single transfer function, when we represent the field by its spectra in the (ω, k) domain. For convenience this transfer function is introduced as a complex-valued length $C^-(\omega, k)$, which may be regarded as the inductive scale-length for the downward diffusing surface field. As outlined in Appendix A, the impedance of the anomalous variation field at the surface of the substratum follows from this scale-length according to

$$\hat{E}_a = i\omega\mu_0 C^- \hat{H}_a^-, \tag{10}$$

which in combination with equation (8b) gives

$$\hat{H}_a^- = \hat{Z}_a/(ikC^-)^{-1} \tag{11}$$

as magnetic boundary conditions for the induction anomaly at the lower surface $z=+0$ of the nonuniform sheet.

With the aid of Wait's recurrence formula in Appendix A, we can derive the transfer function C^- for any given layered substratum. If, for instance, the lower half-space is nonconducting down to the depth $z=h^*$ and perfectly conducting below this depth, we obtain in

$$C^- = k^{-1} \tanh (kh^*) \tag{12}$$

a real transfer function, because inducing and induced field are in-phase. In general, however, $C^-(\omega, k)$ will be complex-valued.

The z-dependence of the anomalous magnetic Fourier spectra $\hat{H}_a$ and $\hat{Z}_a$ in the upper nonconducting half-space is $\exp(|k|z)$, because the anomaly must disappear for $z\to-\infty$ and satisfy the divergence equation (5). Hence, $ik\hat{H}_a+|k|\hat{Z}_a=0$ for $z\leq-0$, yielding

$$\hat{Z}_a = -i \operatorname{sgn}(k)\hat{H}_a^+ \tag{13}$$

as the magnetic boundary condition for the upper surface $z=-0$ of the nonuniform sheet; $\operatorname{sgn}(k)=k/|k|$ denotes the signum function.

The corresponding boundary conditions for the Fourier spectra $\tilde{Z}_a$ and $\tilde{H}_a$ as function of frequency and distance y have the form of convolution integrals (cf. Appendix B). These boundary conditions are obtained by applying an inverse Fourier transformation to the conditions in the (ω, k) domain, which for a function $g(k)$ is defined by

$$f(y) = \int_{-\infty}^{+\infty} g(k) \exp(iky)dk.$$

The convolution theorem for the transform of products yields, when applied to equations (11) and (13), the desired boundary conditions as

$$\tilde{H}_a^- = K^- * \tilde{Z}_a \tag{14}$$

for $z=+0$ and

$$\tilde{Z}_a = K^+ * \tilde{H}_a^+$$

and

$$\tilde{H}_a^+ = -K^+ * \tilde{Z}_a \tag{15}$$

for $z=-0$, with

$$K^+(y) = 1/\pi y \tag{16a}$$

as the inverse Fourier transform of $-i\cdot\operatorname{sgn}(k)$.

$$K^-(\omega, y) = \pi^{-1}\cdot\int_0^\infty \{\sin(ky)/[kC^-(\omega, k)]\}dk \tag{16b}$$

is the inverse Fourier transform of $1/(ikC^-)$. The first transform K^+ is readily identified as Kertz's K-operator (Siebert and Kertz, 1957). The second transform is expressible as a one-sided sine transform, because $C^-(\omega, k)$ is an even function and thus kC^- an odd function of k. If we have, for instance, a perfect conductor at the depth h^* and zero conductivity above, the evaluation of equation (16b) with C^- from equation (12) leads to

$$K^-(y) = \{2h^*\cdot\tanh[(\pi y)/(2h^*)]\}^{-1}.$$

This function is shown by the dashed curve in the upper diagram of Figure 2 for $h^*=181.3$ km.

In order to facilitate numerical calculations of the kernel K^-, we make use of the above cited transform of the signum function and rewrite equation (16b) in the form

$$K^-(\omega, y) = K^+(y) + \pi^{-1}\int_0^\infty \cdot\left[\frac{1}{kC^-(\omega, k)} - 1\right]\sin(ky)dk.$$

The integration can now be terminated at some properly chosen limiting value of k, because the integrand disappears when, for sufficiently large wavenumbers, C^- approaches $1/k$ (cf. Appendix A).

The difference of the kernels K^+ and K^-, as shown in Figure 2 for simple two-layer models, reflects as a function of frequency and distance y

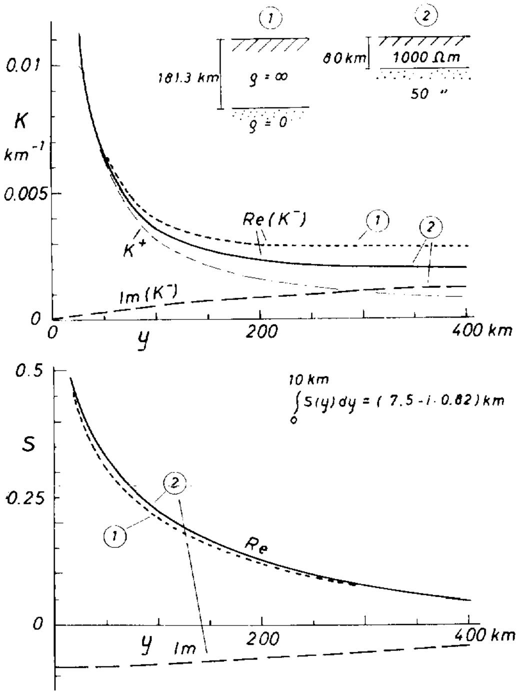

FIG. 2. Upper diagram: Convolution kernels $K^+(y)$ and $K^-(\omega, y)$ connecting the Fourier harmonics of H_a and Z_a as function of y just above and below a thin conducting sheet. The difference between the kernels gives the degree of inductive coupling between sheet and substratum, which, for the chosen substructure and frequency, sets in at about $y = 50$ km distance (cf. text). Lower diagram: Convolution kernel $S(\omega, y)$ connecting the anomalous current in a thin nonuniform sheet with the anomalous part of the electric field [equation (27)]. The complex-valued kernels have been calculated for $f = 1$ cph and the two indicated two-layer models for the conducting substructure.

the changing degree of inductive coupling between sheet and substratum. As $y \to 0$, the real parts of K^- and K^+ merge into the same asymptotic curve, while the imaginary part of K^- goes to zero. Hence, surface anomalies of small lateral extent (in the y direction) are not coupled by mutual induction to the deep conductivity structure, so that $H_a^+ = -H_a^- = j_a/2$. This result applies when the lateral scale-length L of the nonuniform zone is small in comparison with the modulus of the inductive scale-length $C^-(\omega, 0)$ for the substructure. As $y \to \infty$, the kernel K^+ goes to zero, while K^- approaches $1/[2 \cdot C(\omega, 0)]$ as a finite, complex-valued limiting value; this behavior indicates that the maximum possible inductive coupling for extended anomalies occurs when $L \gg |C^-(\omega, 0)|$.

If the conducting material in the lower half-space is replaced by a perfect conductor at the frequency-dependent depth $h^*(\omega) = \mathrm{Re}[C^-(\omega, 0)]$, this replacement gives approximately the same real part of the inductive response for small wavenumbers as the original system. In the case of the two-layer model 2 in Figure 2, the inductive scale-length $C^-(\omega, 0)$ is $(181.3\text{–}108.5\,i)$ km for $f = 1$ cph. By placing the perfect conductor at a depth of 181.3 km (model 1 in Figure 2), we obtain a kernel K^- with the real part of the kernel similar to that for the original model 2. Hence, this often used substitution would yield a useful first approximation for the coupling between sheet and substratum.

INVERSION FORMULA

Once the convolution kernel $K^-(\omega, y)$ has been found for a chosen deep conductivity distribution $\sigma(z)$, we are able to derive without further assumptions the anomalous current distribution $\tilde{j}_a(\omega, y)$ in the (x, y) plane from the anomalous magnetic spectra $\tilde{Z}_a(\omega, y)$ or $\tilde{H}_a^+(\omega, y)$ above this plane, i.e., at the earth's surface. By combining the first field equation (8a), rewritten for the (ω, y) domain, with the boundary conditions (14) and (15), we obtain

$$\begin{aligned} \tilde{j}_a &= \tilde{H}_a^+ - \tilde{H}_a^- \\ &= -[K^+ + K^-] * \tilde{Z}_a \\ &= \tilde{H}_a^+ - K^- * [K^+ * \tilde{H}_a^+]. \end{aligned} \tag{17}$$

The second field equations (3b) and (8b) for the (x, y) plane connect the anomalous electric field with the anomalous behavior in Z independently of the internal conductivity distribution. Both equations are readily transferred into the (ω, y)-domain, yielding

$$\partial \tilde{E}_a / \partial y = i\omega\mu_0 \tilde{Z}_a \tag{18a}$$

and

$$\begin{aligned} \tilde{E}_a &= \tfrac{1}{2} i\omega\mu_0 [G * \tilde{Z}_a] \\ &= \tfrac{1}{2} i\omega\mu_0 [G * (K^+ * \tilde{H}_a^+)], \end{aligned} \tag{18b}$$

with

$$G(y) = \mathrm{sgn}\,(y)$$

as the inverse Fourier transform of $(-2i/k)$. By using equation (31) of Appendix B, the convolution of a function $f(y)$ with the signum function can be written as

$$f * G = \int_0^\infty [f(y-\eta) - f(y+\eta)]d\eta.$$

We now solve the material equation (7b) of the anomalous current density j_a for the unknown perturbation $\tau_a(y)$ and obtain after a transformation into the (ω, y) domain

$$\tau_a = (\tilde{j}_a - \tau_n \tilde{E}_a)/(\tilde{E}_n + \tilde{E}_a). \qquad (19)$$

This formula for the determination of τ_a can be evaluated with an empirical profile of $\tilde{E}_a$, $\tilde{Z}_a$, or $\tilde{H}_a^+$ for one particular frequency. If, for instance, $\tilde{Z}_a(\omega, y)$ is given as a function of y, we derive $\tilde{E}_a$ by integration according to equation (18b) and $\tilde{j}_a$, by convolutions with K^+ and K^- according to equation (17).

There remains the problem of expressing E_n in a similar way in terms of the normal parts of H^+ or Z. The second field equation holds also for the normal field; hence,

$$\tilde{E}_n = \tfrac{1}{2} i\omega\mu_0[G * \tilde{Z}_n] \qquad (20)$$

in analogy with equation (18b). The $(E_n - H_n^+(n))$ impedance relation for the upper surface of the sheet can be established as in the case of the anomalous part (equation 10) by use of a transfer function $C^+(\omega, k)$ for the normal conductivity structure $\sigma(z)$ and τ_n; viz,

$$\hat{E}_n = i\omega\mu_0 C^+ \hat{H}_n^+. \qquad (21)$$

This transfer function is readily derivable from the corresponding scale-length $C^-(\omega, k)$ for the lower $z = +0$ plane, i.e., for the substratum alone, according to

$$C^+ = C^-/(1 + i\omega\mu_0\tau_n C^-) \qquad (21a)$$

as shown in Appendix A. Thus, if $N(\omega, y)$ denotes the inverse Fourier transform of $C^+(\omega, k)$, the transformation of equation (21) into the (ω, y) domain yields

$$\tilde{E}_n = i\omega\mu_0[N * \tilde{H}_n^+], \qquad (22)$$

which is the desired relation between E_n and H_n^+. With the aid of these relations, we are able to express the general inversion formula equation (19) in terms of $\tilde{E}_a$ and $\tilde{E}_n$, $\tilde{Z}_a$ and $\tilde{Z}_n$, $\tilde{H}_a^+$ and $\tilde{H}_n^+$, or in any other combination of one anomalous and one normal part.

Of particular importance for practical applications is the special "Cagniard-case" in which the inducing source field is quasi-uniform in the direction of y. In this case, it can be shown that equation (22) reduces to

$$\tilde{E}_n = i\omega\mu_0 C^+(\omega, 0)\tilde{H}_n^+, \qquad (22a)$$

i.e., the surface impedance is constant along the y axis, while $\tilde{Z}_n$ is zero. Spatial differences in E and H^+ are ascribed, without formal separation, to the anomalous part and variations in Z are considered to be anomalous altogether.

The induction anomaly in this special case is expressible in the form of linear transfer functions, which connect as functions of frequency and distance y the anomalous parts of the field and current distributions with the quasi-uniform normal part of H^+. The transfer functions are denoted by

$$\begin{pmatrix} \tilde{E}_a \\ \tilde{Z}_a \\ \tilde{H}_a^+ \\ \tilde{j}_a \end{pmatrix} = \begin{pmatrix} i\omega\mu_0 c_H \\ z_H \\ h_H \\ q_H \end{pmatrix} \cdot \tilde{H}_n^+ \qquad (23)$$

and can be derived by a spectral correlation analysis of observational data along a profile crossing the anomaly; c_H has the dimensions of length. The inversion formula equation (19) reduces in the Cagniard case to

$$\tau_a(y) = \frac{q(\omega, y)/i\omega\mu_0 - \tau_n c_H(\omega, y)}{C^+(\omega, 0) + c_H(\omega, y)}, \qquad (24)$$

with

$$q_H = -(K^+ + K^-) * z_H$$
$$= -(K^+ + K^-) * (\partial c_H/\partial y),$$
$$c_H = \tfrac{1}{2}[G * z_H],$$

and

$$z_H = K^+ * h_H.$$

A successful interpretation of induction anomalies with the aid of the inversion formulas depends (i) on the proper choice of the normal total conductivity τ_n, (ii) on the proper choice of the deep conductivity distribution $\sigma(z)$, and (iii) on the validity of the assumption that the substructure is without lateral conductivity gradients. On the other hand, a test is provided by the fact that the nonuniformity $\tau_a(y)$, found by inversion, must be real and independent of fre-

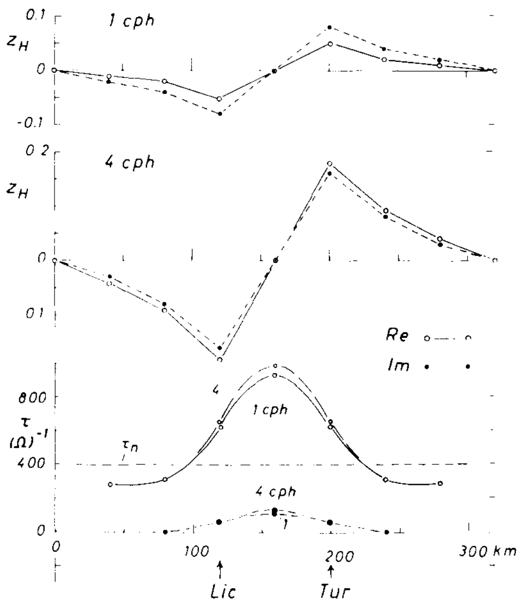

FIG. 3. Application of the inversion formula equation (24) to the induction anomaly of geomagnetic variations in central California. The profiles extend from Monterrey at the Pacific coast in northeasterly direction, crossing the San Joaquin valley between the stations Lick observatory (LIC) and Turlock Lake (Tur). Upper diagrams: Transfer functions z_H for anomalous Z variations after the removal of a coast effect. Lower diagram: Total conductivity found by inversion from the z_H profiles at the two indicated frequencies, using 400 (ohm-m)$^{-1}$ as normal value of τ and model 2 in Figure 2 for the deep conductivity structure. For both frequencies, the inversion yields matching total conductivity anomalies with a negligible imaginary part and with 1000 (ohm-m)$^{-1}$ as maximum total conductivity for the sediments in the center of the San Joaquin valley.

quency. Hence, the interpretation should be carried out with at least two different frequencies.

Figure 3 gives an example of this test. It shows the empirical z_H profiles for two frequencies across the San Joaquin valley in central California; the profiles have been derived from the analysis of anomalous Z variations at six survey stations (Schumucker, 1970; Figure 41). The most prominent feature on these profiles is the coastal anomaly of the Pacific ocean. However, the inland anomaly, which is shown in Figure 3 after the removal of the coast effect, has been well established by a series of cross profiles. The anomaly coincides with the sedimentary basin of the San Joaquin valley, which is bounded by the Coast range to the west and the Sierra Nevada to the east. Because no electric observations have been made, the transfer function c_H for the anomalous part in E has been derived by folding the z_H profile with the signum function G [cf. equation (24)].

The crust and upper mantle are represented by model 2 in Figure 2, which has been found to be appropriate for the interpretation of the coastal anomaly. Hence, the kernel K^- of the same illustration is used to obtain from z_H the normalized anomalous current distribution q_H. Assuming the source field to be quasi-uniform, we get from the recurrence formula of Appendix A as the transfer function $C^+(\omega, 0)$ for the upper surface of the sheet (148.1–120.6 i) km for 1 cph and (77.1–76.2 i) km for 4 cph.

The total conductivity profiles thus derived from equation (24) show a fair agreement for the two frequencies used and are basically real. The total conductivity in the center of the San Joaquin valley is found to be in the order of 1000 (ohm-)$^{-1}$ in comparison to the adopted normal value of 400 (ohm-)$^{-1}$ for the surrounding surface cover. Since the crystalline basement is about 6 km deep near the center of the valley, we obtain 6 ohm-m as mean resistivity for the overlying sediments.

MODEL CALCULATIONS

The inverse problem, namely to find for a given conductivity anomaly $\tau_a(y)$ the anomalous field and current distribution, has been extensively treated by Price (1949) and several other authors (cf. Rikitake, 1966). The following approach employs again a convolution integral to deal with the effect of a conducting substratum.

The anomalous parts of the electric field and current, E_a and j_a, are connected in two ways. The first link gives the material equation (7b),

$$\tilde{j}_a = \tau \tilde{E}_a + \tau_a \tilde{E}_n \quad (25)$$

for the spectra of E_a and j_a as functions of frequency and distance y. The second link follows from the inductive coupling of the anomalous currents in the (x, y) plane with those in the conducting lower half-space. By combining the field equations (8) with the boundary conditions (11) and (13) for the (ω, k) domain, we obtain

$$\hat{E}_a = -\left[i\omega\mu_0 C^- / (1 + |k| C^-)\right] \hat{j}_a \quad (26)$$

or, after an inverse Fourier transformation into the (ω, y) domain,

$$\tilde{E}_a = -i\omega\mu_0[S * j_a]. \tag{27}$$

The convolution kernel

$$S(\omega, y) = \pi^{-1}\int_0^{\infty} \frac{C^-(\omega, k)}{1 + kC^-(\omega, k)} \cdot \cos(ky)dk \tag{28}$$

is the inverse transform of $C^-/(1+|k|C^-)$, which is an even function in k. Hence, the transform can be written as a one-sided cosine transform. It is readily verified that

$$\int_{-\infty}^{+\infty} S(\omega, y)dy = C^-(\omega, 0).$$

Even though the real part of S becomes infinite as $y \rightarrow 0$, the area between $\mathrm{Re}(S)$ and the y axis is finite. The integration in equation (28) for a given substructure and transfer function $C^-(\omega, k)$ can be terminated at some ultimate value $k = k_N$, when the integrand approaches $\cos(ky)/(2k)$, by adding the tabulated cosine integral for the integration between $k = k_N$ and $k = \infty$.

The inductive coupling with a perfect conductor at the depth h^* is given by the kernel

$$S = (4\pi)^{-1} \cdot \ln\left[1 + (2h^*/y)^2\right],$$

as it is readily seen by solving equation (28) with C^- from equation (12). This function is shown in Figure 2, together with the S kernel for a simple two-layer model for the earth's interior.

By inserting equation (27) into equation (25), we obtain

$$\tilde{j}_a = \tau_a\tilde{E}_n - i\omega\mu_0\tau[S * \tilde{j}_a] \tag{29a}$$

as an integral equation for determining $\tilde{j}_a$ as a function of y for a given frequency, conductivity model, and source field configuration. In the case of a quasi-uniform source field, we may replace $\tilde{j}_a$, $\tilde{E}_n$, and $\tilde{E}_a$ by their respective transfer functions as introduced in equation (22a) and equation (23). This gives

$$q_H = i\omega\mu_0[\tau_a C^+(\omega, 0) - \tau(S * q_H)] \tag{29b}$$

as an integral equation for the normalized anomalous current with respect to $\tilde{H}_n^+$. In the case of a nonuniform source field, $\tilde{E}_n$ as a function of position and frequency has to be inferred from the assumed source field geometry and the chosen normal conductivity model without the perturbation $\tau_a(y)$.

Suppose that the kernel S has been calculated for a certain deep conductivity distribution $\sigma(z)$ and that $\tau(y)$ is given at a number of discrete points along the y-axis. This total conductivity profile should have sufficiently long uniform sections on each end and the spacing of the grid points should be small in comparison to the inductive scale length of the substratum. Numerical solutions of the integral equations (29a, b) are then obtained by matrix inversion or by successive substitutions, yielding estimates for the anomalous current density at the chosen grid points.

The iterative solution can be started in two complementary ways similar to Price's (1949) original proposal: (i) We disregard for a first approximation the effect of anomalous self-induction, generating E_a. This procedure gives $j_a^{(1)} = \tau_a E_n$ as a first approximation. By a convolution of $j_a^{(1)}$ with the S kernel for the chosen substructure, we obtain according to equation (29a) a second approximation $j_a^{(2)}$, and so on. The normalized equation (29b) is treated correspondingly.

The convergence of these successive iterations is markedly improved when we include in the first approximation the convolution with S for a small range Δy around the considered point y. As described in Appendix B, we expand $j_a(q_H)$ within this range into a Taylor series, set

$$[S * \tilde{j}_a^{(1)}] \approx 2\tilde{j}_a^{(1)} \int_0^{\Delta y} S(\omega, \eta)d\eta,$$

and obtain in this way

$$\tilde{j}_a^{(1)} = \tilde{E}_n\tau_a \bigg/ \left[1 + 2i\omega\mu_0\tau \int_0^{\Delta y} S(\omega, \eta)d\eta\right]$$

as an improved first approximation.

The anomalous electric field of the last approximation as obtained from equation (25) is then used to derive the anomalous magnetic field according to equations (18a) and (17). The result can be tested with the methods of the previous section. Ideally, an inversion when applied to the calculated induction anomaly should bring back the original total conductivity model $\tau(y)$. The discrepancies indicated in Figure 4 reflect the imperfections of the model calculations and the subsequent inversion with a limited number of grid points.

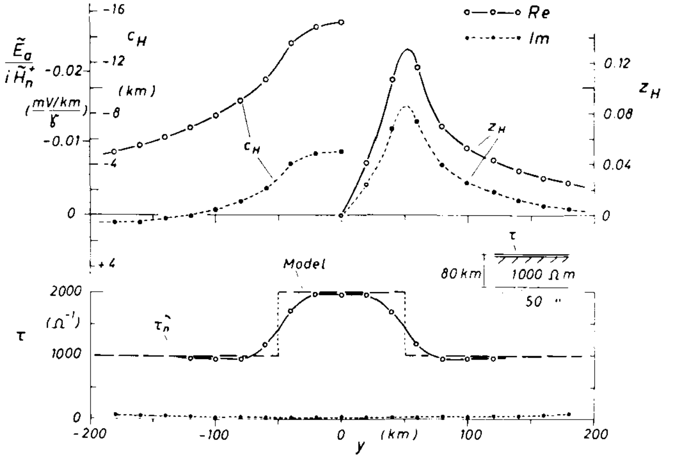

FIG. 4. Model calculations for a conducting thin sheet with a 100 km wide two-dimensional strip of two-fold increased total conductivity. The adopted substructure is model 2 in Figure 2 and the inducing source field (quasi-uniform) is in E polarization with respect to the strip. The transfer functions for the anomalous electric (c_H) and vertical magnetic (z_H) variations have been calculated for a frequency of 1 cph by solving equation (29b) with six iterations and 21 grid points, 20 km apart. The c_H profile is symmetric and the z_H profile mirror-symmetric to the center of the anomaly. The anomaly of the surface impedance can be inferred from the c_H profile, by using the outer left scale [cf. equation (23)]. The inversion of the calculated c_H profile according to equation (24) yields a smoothe image of the original boxcar-type model. The discrepancy is due to the rather limited number of grid points and their coarse spacing.

(ii) If Price's parameter $\beta=\omega\mu_0\tau L$ (L is the half-width of the anomaly) for the anomalous self-induction is larger than unity, we may assume for a first approximation that the anomalous Z variations are completely suppressed. Hence, we set $\tilde{Z}_a^{(1)}=0$ and, therefore, $\tilde{j}_a^{(1)}=0$, implying that $\tilde{E}_a^{(1)}=-(\tau_a/\tau)\cdot\tilde{E}_n$ according to equation (25). The second approximation for $\tilde{Z}_a$ follows by a differentiation of $\tilde{E}_a^{(1)}$ with respect to y [equation (18a)], which in turn leads to improved approximations for $\tilde{j}_a$ [equation (17)] and $\tilde{E}_a$ [equation (25)]. The quality of the final approximation can be tested with the aid of the kernel S in equation (27).

APPENDIX A

CALCULATION OF THE INDUCTIVE SCALE-LENGTH $C(\omega,k)$ FOR A LAYERED CONDUCTING SUBSTRATUM

Let the lower half-space of Cartesian coordinates consist of N uniform conducting layers, the last layer extending downward to infinity; σ_n and d_n shall denote the conductivity and thickness of the nth layer, $n=1, 2, \cdots N$ and $d_N=\infty$. The electromagnetic diffusion equation (6) has in the (ω, k) domain a general solution in terms of hyperbolic functions with

$$K_nd_n = (i\omega\mu_0\sigma + k^2)^{1/2}\cdot d$$

as argument for the nth layer (cf. Wait, 1953). These functions define for each layer a dimensionless ratio

$$g_n = \frac{K_ng_{n+1} + K_{n+1}\tanh(K_nd_n)}{K_{n+1} + K_ng_{n+1}\tanh(K_nd_n)},$$

which can be found by successive substitutions for all layers, beginning with $g_N=1$ (continuity condition). The impedance at the surface of the nth layer is $i\omega\mu_0g_n/K_n$, which defines $C_n=g_n/K_n$ as the inductive scale-length for this surface, including the surface $z=0$. As $k\to\infty$, $K_n\approx k$ for all layers of finite conductivity, i.e., the scale-length C_n approaches $1/|k|$. If the top layer is a "thin layer," we obtain, after replacing the hyperbolic tangent by its argument for $n=1$, equation (21a) with C_1 and C_2 denoting the scale-length just above (C^+) and just below (C^-) of the thin layer.

APPENDIX B

CONVOLUTION WITH THE KERNELS S, K^+, AND K^-

The convolution of a function $A(y)$ with a second function $B(y)$ yields a third function

$$C(y) = A * B = \int_{-\infty}^{+\infty} A(y-\eta)B(\eta)d\eta. \quad (30)$$

If the convolution kernel $A(y)$ is either an even or an odd function of y, the convolution can be written as a one-sided integral:

$$C(y) = \int_0^\infty A(\eta)\cdot[B(y-\eta) \pm B(y+\eta)]d\eta. \tag{31}$$

The upper sign applies when A is even and the lower sign when A is odd.

The real parts of the kernels K^+, K^-, and S [equations (16a), (16b), (28)] approach infinity as $y \rightarrow 0$, i.e., K^- and K^+ have a $1/y$ singularity, while the singularity of S is logarithmic. Therefore, let us expand the function which is to be folded with either one of these kernels within a small neighborhood of y into a Taylor series,

$$B(y \pm \eta) = B(y) \pm \eta B'(y).$$

This gives, when inserted for $\eta \leq \Delta y$ into equation (31),

$$C(y) = 2B(y)\cdot\int_0^{\Delta y} A(\eta)d\eta + \int_{\Delta y}^\infty \cdots d\eta \tag{32}$$

when A is even, and

$$C(y) = -2B'(y)\cdot\int_0^{\Delta y} \eta A(\eta)d\eta + \int_{\Delta y}^\infty \cdots d\eta \tag{33}$$

when A is odd.

The convolution with S is carried out according to equation (32). By reversing the order of integration, we obtain for the numerical evaluation

$$\int_0^{\Delta y} S(\omega, \eta)d\eta = \pi^{-1}\int_0^\infty \frac{C^-(\omega, k)}{k + k^2C^-(\omega, k)} \cdot \sin(\Delta yk)dk.$$

The convolution with K^+ and K^- is carried out according to equation (33) with

$$\int_0^{\Delta y} \eta K^+(\eta)d\eta = \Delta y/\pi.$$

For a sufficiently small distance Δy, the same relation applies also to the kernel K^-.

ACKNOWLEDGMENT

The author is grateful to Dr. P. Weidelt for his advice during the preparation of the manuscript.

REFERENCES

Price, A. T., 1949, The induction of electric currents in non-uniform thin sheets and shells: Quart. J. Mech. and Appl. Math., v. 2, p. 283–310.

Rikitake, T., 1966, Electromagnetism and the earth's interior: Amsterdam, Elsevier Publishing Co.

Schmucker, U., 1970, Anomalies of geomagnetic variations in the southwestern United States: Bull. Scripps Inst. Oceanography, v. 13.

Siebert, M., and Kertz, W., 1957, Zur Zerlegung eines lokalen erdmagnetischen Feldes in äusseren und inneren Anteil: Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse, Abt. IIa, Nr. 5, p. 87–112.

Wait, J. R., 1953, Propagation of radio waves over a stratified ground: Geophysics, v. 18, p. 416–422.