Mr. Dennison's assumptions of a state of complete rest before the arrival of the wave are the more logical ones. I have confirmed his results, solving the problem through the application of the Laplace transform. For completeness, I give here the expression for the response of a receptor to a sine-wave displacement input of amplitude A and frequency $n/2\pi$:

$$\theta(t) = nA\left[(N^2 - n^2)^2 + K^2 n^2\right]^{-1/2}$$

$$\cdot \left\{ n \sin\left[nt - \arc\tan\left(\frac{Kn}{N^2 - n^2}\right)\right] \right.$$

$$\left. + N^2\left(N^2 - \frac{K^2}{4}\right)^{-1/2} e^{K(t/2)} \sin\left[\left(N^2 - \frac{K^2}{4}\right)^{1/2} t + \arc\tan \frac{K\left(N^2 - \frac{K^2}{4}\right)^{1/2}}{N^2 - \frac{K^2}{2}} - \arc\tan \frac{K\left(N^2 - \frac{K^2}{4}\right)^{1/2}}{N^2 - n^2 - \frac{K^2}{2}}\right]\right\}$$

This answer gives the amplitude, θ_0, and phase, ϕ, as expressed in Mr. Dennison's discussion.

COMMENTS ON THE DISCUSSION BY A. T. DENNISON

STEPHEN SZASZ*

Prescott's derivation of the differential equation for θ, which describes the movement of the receptor mass with respect to the frame, is correct (equation (6)); the general solution of this equation (7) is also correct. Dennision also agrees with these equations.

Equation (6) being of second order, its general solution contains two arbitrary constants, θ_0 and ϕ. To find a particular solution applicable to a given physical system, two conditions must be given which, substituted in equation (7), will yield two ordinary equations which determine the numerical values of the arbitrary constants.

Both Prescott and Dennison agree on the first of these initial conditions, namely, $\theta=0$ for $t=0$. They disagree, however, on the proper second condition.

Prescott's second condition, $\theta_0 = A\sqrt{C_1^2+C_2^2}$, states that at $t=0$, the amplitude of the transient term is equal to the amplitude of the steady-state term. No justification is given in physical terms for this condition.

Dennison's second condition is $-\dot{\theta}=\dot{\alpha}$, which means that at $t=0$, the velocity of the receptor mass with respect to a system of coordinates fixed in space is zero. This condition is justified on physical grounds.

From here on, both Prescott's and Dennison's calculations are correct. However, because they describe different systems, it is not surprising that the results are different.

Dennison's criticism, therefore, should be directed not at Prescott's mathematics but at the physical validity of the second initial condition in Prescott's paper.

* Sinclair Research Laboratories, Tulsa, Oklahoma.

A DISCUSSION OF THE "FAULT" AND "DIKE" PROBLEMS IN MAGNETOTELLURIC THEORY

J. T. WEAVER*

In two recent papers appearing in GEOPHYSICS, d'Erceville and Kunetz (1962) and Rankin (1962) have dealt with the magnetotelluric theory for a plane earth which contains a certain type of vertical fault. In both cases the results depend on a boundary condition which requires the assumption that the normal component of current density vanishes at the surface of the earth. While d'Erceville and Kunetz confine their attention to the region below the surface and thereby avoid explicit mention of the source field, Rankin fol-

* Pacific Naval Lab., Victoria, B. C., Canada.

lows Cagniard (1953) by considering a plane-polarized electromagnetic wave normally incident on the surface of the earth. In this case, the assumed boundary condition is not correct, as we shall see later; indeed, it actually leads to a contradiction.

It is the purpose of this note to examine the validity of the boundary condition in some detail, and to point out that if the time variations are considered to be quasi-stationary, then in fact a much stronger assumption is implicit in the form of mathematical model chosen by the above authors.

Throughout this discussion we shall only consider media which have the free space values of permeability and permittivity. In the electromagnetic system of units, this means that the electric displacement is $\boldsymbol{E}/c^2$ (where $\boldsymbol{E}$ is the electric field and c is the velocity of light), and that the magnetic induction is identical with the magnetic field $\boldsymbol{H}$. In addition, all field vectors will be assumed to vary harmonically in time with angular frequency ω, so that all time derivatives may be replaced by $i\omega$ if a factor exp $i\omega t$ is understood throughout. With these simplifications, the field vectors inside a continuous medium of conductivity σ satisfy Maxwell's equations in the form

$$\operatorname{curl} \boldsymbol{H} = \omega^{-1}(\gamma^2 + ik^2)\boldsymbol{E}, \tag{1}$$

$$\operatorname{curl} \boldsymbol{E} = -i\omega \boldsymbol{H}, \tag{2}$$

where $\gamma^2 = 4\pi\sigma\omega$ and $k = \omega/c$. If the field variations are sufficiently slow so that $k^2/\gamma^2 \ll 1$, an approximate form of (1) is

$$\operatorname{curl} \boldsymbol{H} = (\gamma^2/\omega)\boldsymbol{E} \tag{3}$$

which, together with (2) and div $\boldsymbol{H}=0$, yields the diffusion equation

$$(\nabla^2 - i\gamma^2)\boldsymbol{H} = 0. \tag{4}$$

The above approximation fails, even for slowly varying fields, if the medium has negligible conductivity. In this case (1) becomes

$$\operatorname{curl} \boldsymbol{H} = i(k^2/\omega)\boldsymbol{E} \tag{5}$$

from which we obtain the wave equation

$$(\nabla^2 + k^2)\boldsymbol{H} = 0. \tag{6}$$

d'Erceville and Kunetz take the plane $z=0$ as the surface of the earth, with the z-axis directed downwards, and the plane $x=0$ representing a vertical fault, dividing the earth into two regions of conductivity $\sigma_1(x<0)$ and $\sigma_2(x>0)$ respectively. Since the region $z<0$ is occupied by the nonconducting atmosphere, the field vectors there are solutions of the wave equation (6), while inside the earth ($z>0$) they satisfy the diffusion equation (4), where the constant γ has a different value in the two regions of different conductivity. Rankin's model is the same except that in place of a single fault, he considers a dike, of conductivity σ_1, situated between the planes $x = \pm l/2$, the conductivity on either side of the dike being σ_2. Both models also include a common underlying medium of conductivity σ_3, but we shall disregard it in this discussion since its presence has no effect on the arguments to be presented here.

In addition, two simplifying assumptions are made by these authors. They are (1) all quantities are independent of the variable y, and (2) the magnetic vector is everywhere in the y-direction. Under these conditions, and with $\boldsymbol{i}$, the current density, introduced through the relation $\boldsymbol{i} = \sigma\boldsymbol{E}$, the z-component of equation (3) becomes

$$\partial H_y/\partial x = 4\pi i_z.$$

If it is now assumed that $i_z=0$ at $z=0$, it follows immediately that H_y is constant on $z=0$, despite the change of conductivity at a discontinuity.

However, this conclusion contradicts the results obtained by considering a plane wave normally incident on $z=0$. To show this, we consider first the plane-polarized wave

$$H_y^r = Ae^{-ikz}, \qquad E_x^r = Ace^{-ikz}$$

incident on a *uniform* conducting earth. This gives rise to a reflected wave

$$H_y^r = Be^{ikz}, \qquad E_x^r = -Bce^{ikz},$$

and a transmitted field which, being a solution of (4), can be written in the form

$$\overset{t}{H_y} = Ce^{-\gamma z\sqrt{i}}, \qquad \overset{t}{E_x} = (\omega\sqrt{i}/\gamma)Ce^{-\gamma z\sqrt{i}}.$$

Fulfillment of the boundary conditions specifying the continuity of H_y and E_x at $z=0$ requires

$$B = \frac{A(1 - k\sqrt{i}/\gamma)}{1 + k\sqrt{i}/\gamma} = A(1 - 2k\sqrt{i}/\gamma) + O(k^2/\gamma^2)$$

and

$$C = 2A/(1 + k\sqrt{i}/\gamma) = 2A(1 - k\sqrt{i}/\gamma) + O(k^2/\gamma^2).$$

Since the terms $O(k^2/\gamma^2)$ are negligible compared with unity, the magnetic field at the surface of the earth is given by

$$[H_y]_{z=0} = 2A(1 - k\sqrt{i}/\gamma). \tag{7}$$

Now in considering the vertical fault model, d'Erceville and Kunetz assume that the solutions for the magnetic fields in their respective regions within the earth can be written in the form

$$H_i(z) + P_i(x, z), \tag{8}$$

where H_i is the solution for a uniform earth of conductivity σ_i, and P_i is a perturbation term tending to zero with increasing distance from the fault. (The subscript, i, takes the value 1 or 2 depending on the region referred to.) If it is now assumed that these fields are due to a plane wave normally incident on the earth, it follows from (7) that the total magnetic field on $z=0$ is

$$2A(1 - k\sqrt{i}/\gamma_i) + P_i(x, 0). \tag{9}$$

Since $P_i \to 0$ as $x \to \pm\infty$, we have from (9),

$$[H_y]_{z=0} \to 2A(1 - k\sqrt{i}/\gamma_1) \quad \text{as} \quad x \to -\infty$$

and

$$[H_y]_{z=0} \to 2A(1 - k\sqrt{i}/\gamma_2) \quad \text{as} \quad x \to +\infty,$$

and since $\gamma_1 \neq \gamma_2$, these results are clearly incompatible with the requirement that H_y be constant on $z=0$.

Actually, the solutions of d'Erceville and Kunetz are derived without any consideration of the field external to the earth, and are therefore merely dependent on the validity of the boundary condition that $i_z=0$ at $z=0$. But this, as we have seen, will automatically exclude the possibility of plane wave incidence as an exciting mechanism. On the other hand, Rankin does consider a plane wave to be incident on his dike model, so that his use of the above boundary condition represents an inconsistency in his theoretical development. Moreover, he also assumes that the total magnetic field can be written in the form (8), but then states that the field on $z=0$ is

$$H_0 + P_i(x, 0),$$

where H_0 is a constant. This is clearly inconsistent with (9) since the first term there is dependent on γ_i and therefore on conductivity. By applying the above boundary condition (which implies that the total magnetic field is constant on $z=0$), and using the fact that $P_i \to 0$ as $x \to \pm\infty$, Rankin then deduces that P_i vanishes on the surface. Actually, however, it can be seen from (9) that it is the functions

$$P_1(x, 0) = 2Ak\sqrt{i}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right), \qquad P_2(x, 0) = 0$$

which meet these requirements.

It is not difficult to see why a contradiction is introduced. Rankin states that "at any horizontal surface on the other side of which the conductivity is zero, we have $i_z=0$," but this is not necessarily correct, since physically it is possible for such a current to exist, causing an oscillation of surface charge density at the interface. In fact, it is readily shown (Weaver, 1962) that the magnitude of i_z at $z=0$ is of the same order as the right-hand term in equation (5), i.e., the displacement current in the *nonconducting* region above the earth. Now if one is considering electromagnetic wave propagation in this region, it is essential to retain the displacement current term, and, for consistency, all other terms of like magnitude as well. We conclude therefore, that it is incorrect to put $i_z=0$ at $z=0$ in a problem involving electromagnetic waves incident on the earth. Conversely, making i_z vanish at $z=0$ implies the neglect of displacement currents above the earth, and then we can no longer speak of electromagnetic wave propagation there.

Under certain circumstances it is in fact possible to neglect the displacement current in a nonconducting medium. Such is the case if the wavelength of the field is much larger than the dimensions of the region under consideration, for then we are concerned only with points sufficiently close to the source of the electromagnetic field so that it is the induction field, rather than the radiation field, which predominates. In this case (5) reduces to

$$\text{curl}\, \boldsymbol{H} = 0 \tag{10}$$

and the problem becomes quasi-stationary.

Now it is generally believed that the natural oscillating electromagnetic field employed in magnetotelluric methods arises from current systems located in the ionosphere. Moreover, the frequencies involved are sufficiently low that the surface of the earth is much less than a wavelength's distance away from the current sources. Thus it is possible, in general, to use the quasi-stationary approximation (10) and to dispense entirely with the concept of wave propagation.

Because of the simplifying assumptions (i) and (ii) made in setting up the "fault" and "dike" models, equation (10) reduces to the pair of equations

$$\partial H_y/\partial z = \partial H_y/\partial x = 0$$

when written in component form. It follows at once that (for quasi-stationary fields) H_y is constant throughout the whole region $z \leq 0$, and not only at the surface $z=0$. Thus, under those conditions when the normal component of current density at the surface of the earth can be regarded as negligible, it is actually implicit in the form of mathematical model chosen that the total (inducing plus induced) magnetic field is constant *everywhere* above the earth. This is a much stronger assumption than is immediately apparent from reading the papers of d'Erceville and Kunetz and of Rankin, but it is a necessary one if their results are to remain valid.

We conclude, therefore, that only if the magnetic field above the earth is quasi-stationary and uniform are the solutions of the above authors correct. Further to this, it must be pointed out that the uniformity requirement is a very restrictive one, which considerably limits the usefulness of the results in applications. In fact, the theory of Cagniard (1953) was criticized on this very point by Wait (1954). He showed that the results based on a uniform field assumption would only be applicable to situations in which the source field originated with vast ionospheric current sheets whose horizontal dimensions were larger than the "skin depth" in the ground. More recently, Price (1962) has indicated that if the conductivity is taken to vary with depth (as is done, in fact, in the models of d'Erceville and Kunetz, and Rankin), then a uniform field assumption can yield very inaccurate results even if the ionospheric currents are of global dimensions. It seems, therefore, that some caution must be exercised in trying to apply the theoretical solutions for the "fault" and "dike" models to a practical situation.

REFERENCES

Cagniard, L., 1953, Basic theory of the magneto-telluric method of geophysical prospecting: Geophysics, v. 18, pp. 605–635.
d'Erceville, I., and Kunetz, G., 1962. The effect of a fault on the earth's natural electromagnetic field: Geophysics, v. 27, pp. 651–665.
Price, A. T., 1962, The theory of magneto-telluric methods when the source field is considered: J. Geophys. Res., v. 67, pp. 1907–1918.
Rankin, D., 1962, The magneto-telluric effect on a dike: Geophysics, v. 27, pp. 666–676.
Wait, J. R., 1954, On the relation between telluric currents and the earth's magnetic field: Geophysics, v. 19, pp. 281–289.
Weaver, J. T., 1962, A note on the vertical fault problem in magneto-telluric theory: Pacific Naval Laboratory Tech. Memo. 62-1, Victoria, B. C.

REPLY OF MESSRS I. D'ERCEVILLE* AND G. KUNETZ* TO THE DISCUSSION BY J. T. WEAVER

The authors have read Mr. Weaver's remarks with great interest. They would like to point out that their aim has not been to clear up the still debated question of how to define the primary field that generates the telluric currents, but to study the effect of a fault on a special type of field, complying with the laws of electromagnetism (neglecting the displacement currents, as is usual at the frequences considered) and differing little from actual telluric currents.

Moreover, the results show that the effect of a fault has a rather limited extension, so that in this case "the infinite is quite near." To get valid results, it will then be sufficient that the telluric current complies with the type considered in an area near the observation point.

* Compagnie Generale de Geophysique, Paris, France.

REPLY BY D. RANKIN* TO THE DISCUSSION BY J. T. WEAVER

I am indebted to Weaver if he has indeed clarified certain points which I had previously considered to be obvious. Cagniard (1953) states explicitly the magnitude of the wavelengths in free space and it is further implicit in the work of Rankin (1962) that it is indeed this same electromagnetic field which is being considered. The plane wave aspect of the problem arises from the *extent of* and not the *distance from* the source so that truly it is the induction field and not the radiation field that is under discussion. I had believed, until this note by Weaver, that d'Erceville and Kunetz (1962) also considered a plane wave incident on the earth and in fact that I was merely following both Cagniard and d'Erceville and Kunetz in this matter. The consistency of the results would tend to confirm this belief.

The last two formulae of the appendix in the work of Rankin (1962) give explicitly the approximation which is implicit throughout the paper and I believe also in d'Erceville and Kunetz's work. Comparison with Weaver's formulae:

$$[Hy]_{z=0} \rightarrow 2A(1 - k\sqrt{j\gamma_1}) \quad \text{as} \quad x \rightarrow -\infty$$

and

$$[Hy]_{z=0} \rightarrow 2A(1 - k\sqrt{j\gamma_2}) \quad \text{as} \quad x \rightarrow +\infty$$

shows that

$$[Hy]_{\substack{z=0\\x=-\infty}} = [Hy]_{\substack{z=0\\x=+\infty}} .$$

In this same approximation $i_z=0$ and the consequence that H_y is constant across the trace of a fault or dike thus follows even as in the reformulation by Weaver. An experimental verification of this result is contained in a forthcoming paper by the author.

REFERENCES

Cagniard, L., 1953, Basic theory of the magnetotelluric effect: Geophysics, v. 18, pp. 605–635.
d'Erceville, I., and Kunetz, G., 1962, The effect of a fault on the earth's natural electromagnetic field: Geophysics, v. 27, pp. 651–665.
Rankin, D., 1962, The magnetotelluric effect on a dike: Geophysics, v. 27, pp. 666–676.

* Physics Department, Victoria College, Victoria, B. C., Canada.