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The Inverse Problem of Electromagnetic Induction: Existence and Construction of Solutions Based On Incomplete Data

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A theory is described for the inversion of electromagnetic response data associated with one-dimensional electrically conducting media. The data are assumed to be in the form of a collection of (possibly imprecise) complex admittances determined at a finite number of frequencies. We first solve the forward problem for conductivity models in a space of functions large enough to include delta functions. Necessary and sufficient conditions are derived for the existence of solutions to the inverse problem in this space. The approach relies on a representation of real-part positive functions due to Cauer and an application of Sabatier's theory of constrained linear inversion. We find that delta-function models are fundamental to the problem. When existence of a solution has been established for a given set of data, actual conductivities fitting the measurements may be explicitly constructed for various special classes of functions. For a solution in delta functions or homogeneous layers a development as a continued fraction is the essential idea; smoothly varying models are found with an adaption of Weidelt's analytic solution.

INTRODUCTION

When long-period fluctuating magnetic fields impinge upon the earth, they diffuse into the deep interior. A combination of observable electromagnetic fields at the surface is controlled by the electrical conductivity inside the earth and is quite independent of the physical mechanism causing the magnetic fluctuations. Therefore measurements of this quantity, which we shall call the admittance, can in principle supply information about the conductivity structure of the earth; the task of extracting the information is the electromagnetic induction inverse problem. The particularly simple variant in which the conductivity is assumed to vary only with depth (or radius in a spherical model) has received a great deal of attention in the geophysical literature, but it would be fair to say that no entirely satisfactory solution has yet been found for the practical case with imprecise and incomplete data. Considerable progress has been made, however, in the solution of an idealized problem where the admittance is assumed to be known exactly at all frequencies: the work of *Weidelt* [1972] and *Bailey* [1970] provides answers to the questions of construction and uniqueness of solutions in certain classes of smoothly varying conductivity models. Weidelt's work lays the foundation for the investigations described here.

In this paper we present solutions to the problems of existence and construction when the admittance data consist of a finite collection of (possibly imprecise) complex numbers, which is the form that results from time-series analysis of the actual field measurements. Simply put, this means that we can determine whether a particular set of data is compatible with the mathematical model of electromagnetic induction in a one-dimensional conductivity structure, and if it is, we can produce a profile fitting the data. Each observation may be exact or it may be specified as lying within a definite range of values; alternatively, the measurements may be associated with estimates of their statistical uncertainty, and in that case it is ascertained whether a solution exists at a specified confidence level. When the data set is compatible with the mathematical model, explicit conductivity profiles can be constructed fitting the data in the appropriate way. It is important to understand that the method of construction is not by trial and error but that it is a direct solution whose accuracy is limited only by the resolution of the discrete representation in the computer and by the precision of the computer arithmetic.

The key to these results is an extension of the class of admissible conductivity profiles. Elementary differential equation theory and most work on geophysical inverse problems restrict attention to spaces of relatively smooth functions; for example, Weidelt's conductivity profiles are confined to those that are continuously differentiable with respect to depth, and other treatments require at least piecewise continuity. We find that the natural setting for studies with finite data sets is a space permitting delta functions in the conductivity. First we show that the forward problem always has solutions for any element in an appropriately large space of models. Then we obtain a representation of the admittance data as linear functionals with convex constraints. This enables us to apply the infinite-dimensional form of linear programing developed by *Sabatier* [1977] and thereby to derive necessary and sufficient conditions for the existence of solutions to the inverse problem. It is found that if there are any solutions to the problem, there must be one in terms of delta functions, a result that indicates how central the role of distributions is to this problem. Finally, we consider means of explicitly calculating profiles: this can only be done for certain special classes of conductivity functions including delta-function models and continuously differentiable models. When the data are imprecise, the best-fitting model in the least-squares sense can be found by quadratic programing techniques, and it too consists of a finite number of delta functions in conductivity.

PRELIMINARIES

We study electromagnetic induction by a uniform horizontal magnetic field with periodic behavior in time $e^{i\omega t}$; the conductivity σ varies only with z, the vertical coordinate (positive upward); the conducting layer lies between $z = h$ at the surface and $z = 0$ at the bottom, where there is a perfect conductor. *Weidelt* [1972] has shown that the equations for this system can be readily transformed to give solutions to the problem of global induction in a

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spherical earth or that of induction over a half space with horizontally varying fields. His notation will be followed approximately in the initial development. The electric field is horizontal and is perpendicular to the magnetic field. The differential equation for the complex electric field $E(z, \omega)$ in the medium is

$$\frac{\partial^2 E}{\partial z^2} = i\omega\mu_0\sigma E \tag{1}$$

with boundary conditions $E(0, \omega) = 0$, $\partial E/\partial z|_0 = E_0' \neq 0$. Measurements are made of the electric and magnetic fields $E(h, \omega)$, $B(h, \omega)$ as functions of frequency. We define the ratio

$$c(\omega) = \frac{E(h, \omega)}{i\omega B(h, \omega)} = \frac{E(h, \omega)}{\partial E/\partial z|_h} \tag{2}$$

to be the admittance. In the inverse problem it is assumed that c is known at a finite number of frequencies:

$$c_j = c(\omega_j) \quad j = 1, 2, \cdots N \tag{3}$$

or that other (e.g., statistical) information is available about c at these frequencies.

As stated in the introduction, we will need conductivity profiles consisting of delta functions as well as ordinary functions. Delta-function conductivities are already familiar to geophysicists through the work of *Price* [1949] and they have been introduced in a primitive manner into the inverse problem [*Parker,* 1972]. It is convenient to be able to deal with pathological functions and regular ones in a unified way so as to be able to define a distance between any pair of conductivities, for example. Probably more important, a general treatment enables us to demonstrate that an admittance is defined for any model that we might encounter. Indeed, a proper theory of existence must be based in a well-defined class of models. Only fairly elementary methods will be used throughout; the reader will find any unfamiliar mathematical terms clearly explained in *Korevaar's* [1968] excellent book.

We introduce the Banach space $NBV(0, h)$ of real functions of bounded variation on the interval $(0, h)$; for every element $\tau \in NBV$, $\tau(0) = 0$, and the norm is defined by

$$||\tau|| = V_0^h[\tau]$$

the total variation of τ on the interval and every valid element must have a finite norm. This defines a complete normed vector (Banach) space with Stieljes integrals as linear functionals:

$$F[\tau] = \int_0^h f(z)\, d\tau(z)$$

where f is a (possibly complex valued) continuous function. Then we have

$$|F[\tau]| \leqslant |f|_{max} \cdot ||\tau||$$

In our application, τ is to be thought of as the conductivity integrated from the bottom, so that if σ is an ordinary integrable function,

$$\tau(z) = \int_0^z \sigma(y)\, dy$$

Defined this way, τ is of course continuous, but more generally, it will exhibit discontinuities; in particular, a simple jump in τ corresponds to a delta function in conductivity. When $\tau \in NBV$, we say $\sigma \in S$. Since $\sigma \geqslant 0$ in physical systems, all realizable τ are nondecreasing functions, and we then say $\sigma \in S^+$; the set S^+ is called a positive cone in S [*Luenberger,* 1969, p. 214]. If, in addition, σ is an ordinary function, the norm of τ is simply the integral of σ through the layer.

The forward problem must be entirely reformulated in terms of τ. In place of (1) we obtain an integral equation by integrating twice and integrating by parts; then the remaining integral is replaced by the general linear functional over NBV, a Stieljes integral:

$$E(z, \omega) = E_0 + zE_0' + i\omega\mu_0 \int_0^z (z - y)\, E(y, \omega)\, d\tau(y) \tag{4}$$

Here E_0 and E_0' are identified as the boundary conditions applied at $z = 0$, which will be left arbitrary for the time being. The derivative $\partial E/\partial z$ is not defined at every point; therefore we must define one by integration of (1):

$$E'(z, \omega) = E_0' + i\omega\mu_0 \int_0^z E(y, \omega)\, d\tau(y) \tag{5}$$

which is defined everywhere. Obviously, for ordinary conductivities, (4) is equivalent to (1) with the corresponding boundary conditions. Now we show that (4) has a solution for every $\tau \in NBV$, that is, $\sigma \in S$.

Rewrite (1) in its double-integral form:

$$E(z, \omega) = f(z) + \nu \int_0^z dy \int_0^y E(x, \omega)\, d\tau(x)$$

or

$$E = f + \nu\, T[E] \tag{6}$$

where $f(z) = E_0 + zE_0'$, which is clearly continuous, and $\nu = i\omega\mu_0$. Imagine for the moment that ν is fixed; a solution to (6) is developed by successive approximations in the usual way [*Riesz and Sz.-Nagy,* 1965, p. 146]:

$$\tilde{E} = f + \sum_{n=1}^{\infty} \nu^n\, T^n[f] \tag{7}$$

where T^n denotes n applications of the linear operator T. First we show the series converges for every finite ν.

$$|T[f](z)| = \left|\int_0^z dy \int_0^y f(x)\, d\tau(x)\right|$$

$$\leqslant \int_0^z dy \left|\int_0^y f\, d\tau\right|$$

$$\leqslant \int_0^z |f|_{max} \cdot ||\tau||\, dy = z\, |f|_{max} \cdot ||\tau||$$

By repetition of this reasoning it is easily seen that

$$|T^n[f](z)| \leqslant \frac{z^n}{n!} ||\tau||^n \cdot |f|_{max}$$

Thus (7) is majorized by the series

$$\sum_{n=1}^{\infty} \frac{(z|\nu| \cdot ||\tau||)^n}{n!} |f|_{max}$$

which is absolutely and uniformly convergent for all finite z [*Whittaker and Watson,* 1962, p. 581]. Hence by the M test, (7) is uniformly convergent, and therefore by substituting $\tilde{E}$ into (6) we verify that it is a solution. Each of the terms in (7) is absolutely continuous in z; it follows from the uniform convergence of the series that E is a continuous (even an absolutely continuous) function of z.

The above arguments have considered $\nu = i\omega\mu_0$ to be fixed. With z fixed instead, we see that (7) also represents a Taylor series in ω for $E(z, \omega)$, which is convergent in the finite complex ω plane; thus E is an entire function of ω. By substituting the series solution (7) into (5) it is easily seen that E' is an entire function of ω as well. From this it follows that the only singularities of c are those at the zeros of $E'(h, \omega)$ and a possible essential singularity at infinity. The singularities may be characterized much more sharply as follows: consider

$$I(\omega) = \int_0^h E'(y, \omega)E'^*(y, \omega)\, dy + i\omega\mu_0 \int_0^h E(y, \omega)E^*(y, \omega)\, d\tau(y) \quad (8)$$

For $\sigma \in S^+$ it is clear $I(\omega)$ cannot vanish unless ω is on the positive imaginary axis. Because E is an absolutely continuous function of z the first term may be written as a Stieljes integral [*Korevaar*, 1968, p. 411], and (5) may be used to define $dE'(y)$; then

$$I(\omega) = \int_0^h E'\, dE^* + \int_0^h E^*\, dE'$$

The first term is integrated by parts [*Korevaar*, 1968, p. 409], and the boundary condition that $E_0 = 0$ is then applied; this yields

$$I(\omega) = E'(h, \omega)\, E^*(h, \omega)$$

This shows that there are no singularities (or zeros) of c unless ω is on the positive imaginary axis.

There is still more to be learnt from $I(\omega)$. Obviously, from (2),

$$c(\omega) = I^*(\omega)/|E'(h, \omega)|^2$$

Together with (8) this shows that if ω is in the lower half-plane ($\text{Im}\, \omega \leqslant 0$), the real part of $c(\omega)$ is positive, and the imaginary part is negative in the right half-plane ($\text{Re}\, \omega > 0$). The theory of real-part positive functions plays an important role in the study of electrical networks [*Wohlers*, 1969], and we draw upon that powerful theory. The critical result for our purposes is a theorem of *Cauer's* [1932] illustrated extensively by *Wohlers* [1969, pp. 48-62]: for any analytic function g of a complex variable ω with the property that $\text{Re}\, g(\omega) > 0$ when $\text{Re}\, \omega > 0$,

$$g(\omega) = \int_{-\infty}^{\infty} \frac{1 - i\omega\lambda}{\omega - i\lambda}\, db(\lambda) + ib_0 + \omega b_1 \quad \text{Re}\, \omega > 0$$

where b is a real non-decreasing function of bounded variation on the real interval $-\infty < \lambda < \infty$ and b_0, b_1 are real constants with $b_1 \geqslant 0$. To be consistent, ω should be dimensionless in this representation; alternatively, a constant with dimensions of frequency squared should replace the one in the numerator. To apply the representation, let $g = ic$. Then

$$c(\omega) = \int_{-\infty}^{\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega}\, db(\lambda) + b_0 - i\omega b_1 \quad \text{Re}\, \omega > 0$$

For our admittances the representation can be somewhat simplified as follows. The lower limit of the integral can be made zero. To see this, note that unless the contribution from $\lambda < 0$ vanishes identically (i.e., $b(\lambda)$ is constant when $\lambda < 0$), $c(\omega)$ would not have a positive real part everywhere in the lower half-plane. The symmetry $c(-\omega^*) = c^*(\omega)$ shows that this representation holds in the left and right half-planes and, by analytic continuation, also on the negative imaginary axis. Further, c has only poles on the positive imaginary axis, and therefore b has all its variation at jumps that correspond to simple poles in c with positive residues [*Wohlers*, 1969, p. 53]. We can also show that b_1 is zero. For large $|\omega|$ the integral is $O(|\omega|^{-1})$, and therefore, unless b_1 vanishes, c will grow without bound. Physically, $|c(\omega)|$ is the apparent skin depth for fields varying with frequency ω, so that we should expect c to decrease with frequency. A proof of this can be constructed by considering $c(-i\gamma)$, where γ is real and positive; then c is real, and by an argument given by Weidelt (appropriately extended to $\sigma \in S^+$) we can show $dc/d\gamma < 0$, that is, c decreases down the negative imaginary ω axis. Since $c > 0$, we must conclude that c is bounded and so $b_1 = 0$. Thus we have shown that for any $\sigma \in S^+$ the admittance c may always be written

$$c(\omega) = b_0 + \int_0^{\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega}\, db(\lambda) \quad (9)$$

except at the poles of c on the positive imaginary axis.

Finally we show the connection of (9) with other more familiar representations. Weidelt gives a Mittag-Leffler expansion whose integral form is

$$c(\omega) = \int_0^{\infty} \frac{da(\lambda)}{\lambda + i\omega} \quad (10)$$

The relationship to (9) is clear if we perform a simple rearrangement

$$c(\omega) = b_0 + \int_0^{\infty} \left\{ \frac{1 + \lambda^2}{\lambda + i\omega} - \lambda \right\} db(\lambda) \quad (11)$$

Provided the integrals of the two terms in braces converge separately (which is not in general guaranteed), the two equations correspond, except for different measure functions and the absence of a constant term in (10). There is no constant in (10) because such terms arise from a zero-conductivity layer at the surface, which is forbidden in Weidelt's treatment.

The infinite-sum form of (10) can be obtained as an eigenfunction expansion of (1). Suppose $\sigma > 0$ and that it is twice differentiable. Consider the eigenvalue problem

$$\frac{1}{\mu_0\sigma} \frac{\partial^2 u_n}{\partial z^2} = -\lambda_n u_n$$

with boundary conditions $u_n(0) = u_n'(h) = 0$. Also consider $G(z, z_0)$, the Green's function for the inhomogeneous form of (1) with the same boundary conditions:

$$\frac{\partial^2 G}{\partial z^2} - i\omega\mu_0\sigma G = \delta(z - z_0)$$

It can be shown [*Parker*, 1977] that $c(\omega) = -G(h, h)$. But G can also be expanded in u_n, which form an infinite set of orthonormal functions complete on $L^2[0, h]$:

$$G(z, z_0) = \frac{-1}{\mu_0\sigma(z_0)} \sum_{n=1}^{\infty} \frac{u_n(z)u_n^*(z_0)}{\lambda_n + i\omega}$$

Hence

$$c(\omega) = \sum_{n=1}^{\infty} \frac{|u_n(h)|^2/\mu_0\sigma(h)}{\lambda_n + i\omega}$$

Thus the points of discontinuity of b are the eigenvalues of (1), and even in the general problem we shall refer to this set of points as the spectrum of the system. Spectral expansions like this are used a great deal in the general theory of scattering [*Newton*, 1966], of which electromagnetic induction is a simple example [*Raiche*, 1974].

EXAMPLES

A few special types of conductivity profile constitute all the important cases of geophysical and theoretical interest. The most fundamental to this paper is the subset of models $D^+ \subset S^+$, in which σ consists of a finite comb of positive delta functions. More precisely, $\sigma \in D^+$ if the corresponding τ is of the form

$$\tau(z) = \sum_{l=0}^{k} \tau_l \quad z_k \leqslant z < z_{k+1}$$

where $\tau_0 = 0$, $\tau_1, \tau_2, \cdots \tau_K > 0$, and $z_0 = 0$, $0 < z_1 < z_2 \cdots < z_K \leqslant h$. In familiar notation,

$$\sigma(z) = \sum_{l=1}^{K} \tau_l \, \delta(z - z_l)$$

This type of model is the exact analog of the beads-on-a-string system used by *Krein* [1952] and *Barcilon* [1975] to study the eigenvibration problem. From (4) and (5) with $z_k \leqslant z \leqslant z_{k+1}$, we see that E is in the form $f_k + zg_k$, where f_k, g_k are complex constants depending on frequency. Define $C_k(\omega) = E(z_k, \omega)/E'(z_k, \omega)$; then

$$C_k = z_{k+1} - z_k + \frac{1}{i\omega\mu_0\tau_k + 1/C_{k-1}}$$

Upon concatenation of expressions like these for every interval we arrive at the continued fraction solution to the forward problem:

$$c(\omega) = h - z_K + \cfrac{1}{i\omega\mu_0\tau_K + \cfrac{1}{z_K - z_{K-1} + \cfrac{1}{i\omega\mu_0\tau_{K-1} + \cdots \cfrac{1}{z_1 - z_0}}}} \tag{12}$$

Rationalized, the continued fraction represents a function of ω that is the ratio of two polynomials of degree K, so that there are exactly K poles of c on the positive imaginary ω axis corresponding to the K points of discontinuity of b; hence the system has a finite spectrum. Conversely, it will be shown that when b consists of only a finite number of jumps, $\sigma \in D^+$. Because (9) comprises a finite sum of contributions from the discontinuities in b, a finite expansion similar to Weidelt's form exists:

$$c(\omega) = a_0 + \sum_{n=1}^{K} \frac{a_n}{\lambda_n + i\omega} \qquad a_0 \geqslant 0,\ a_n > 0 \tag{13}$$

A quite different analog for $\sigma \in D^+$ is the electrical ladder network shown in Figure 1; the impedance of that network is

$$\mathbf{Z}(\omega) = \mathbf{R}_{K+1} + \cfrac{1}{i\omega\mathbf{C}_K + \cfrac{1}{\mathbf{R}_K + \cfrac{1}{i\omega\mathbf{C}_{K-1} + \cdots \cfrac{1}{\mathbf{R}_1}}}}$$

Comparing this continued fraction with (12), we see that the resistances $\mathbf{R}_k$ correspond to the separations between the conducting sheets and the capacitances $\mathbf{C}_k$ to the integrated conductances $\mu_0\tau_k$.

When the conductivity profile is an ordinary function, the electrical analog is a 'distributed' RC network rather than a 'lumped' element network (the relationship to transmission lines has been noted before; see *Madden and Swift* [1969], for example). We now examine a class of conductivity profiles chosen in such a way that the theory of transmission line synthesis can be borrowed. In geophysics a popular model for modeling almost any one-dimensional system has been a stack of homogeneous layers; here we take a finite pile of uniform slabs of conductivities σ_k and thicknesses h_k such that the product $\sigma_k h_k^2$ is constant. We name the set of conductivity models of this type H^+. Seismologists have used similar structures in which the travel-time is the same in each of a series of homogeneous layers [*Goupillaud,* 1961]; note that the dimensions of the product $\mu_0\sigma_k h_k^2$ are those of time. Consider the admittance C_k measured at the top of the kth slab (the one with conductivity σ_k). Application of the standard boundary conditions that E and $\partial E/\partial z$ be continuous (because τ is a continuous function for $\sigma \in H^+$) yields

$$C_k = \frac{\sinh \Omega_k h_k + \Omega_k C_{k-1} \cosh \Omega_k h_k}{\Omega_k \cosh \Omega_k h_k + \Omega_k^2 C_{k-1} \sinh \Omega_k h_k}$$

where $\Omega_k = (i\omega\mu_0\sigma_k)^{1/2}$. Following the treatment of distributed RC lines given by *Ghausi and Kelly* [1968, p. 213], we define $d = (\mu_0\sigma_k h_k^2)^{1/2}$, which is constant throughout the system, and $P = \cosh d(i\omega)^{1/2}$, which depends only on d and the frequency. Also we introduce a modified admittance:

$$\tilde{C}_k = [d(i\omega)^{1/2} \sinh d(i\omega)^{1/2}] \cdot C_k$$

and we define $\tilde{c}$ to be the admittance at the top of the system, so that $\tilde{c} = \tilde{C}_K$. After a little algebra we obtain

$$\tilde{C}_k = h_k P - \frac{h_k}{P + \tilde{C}_{k-1}/d} \tag{14}$$

Clearly this relationship can be used to build a continued fraction similar to (12), in which P plays the part of ω. A representation like (13) exists, but because of the properties of $\tilde{c}$ it takes a special form: *Ghausi and Kelly* [1968, p. 215] show

$$\frac{\tilde{c}}{P^2 - 1} = \frac{l_1}{P - 1} + \frac{l_2}{P + 1} + \sum_{k=1}^{K} \frac{q_k}{P - p_k} \tag{15}$$

where $l_1, l_2, q_k \geqslant 0$, and $-1 < p_k < +1$. This approach to RC transmission line synthesis is called *O'Shea's* [1965] transformation. The perfect conductor at the bottom of the layer is equivalent to a short-circuit termination of the line, which has two important consequences: the constants p_k occur in pairs $+p_k, -p_k$, each associated with the same q_k; and the coefficients l_1, l_2 both vanish. These facts are most easily ascertained from (14) directly. From the con-

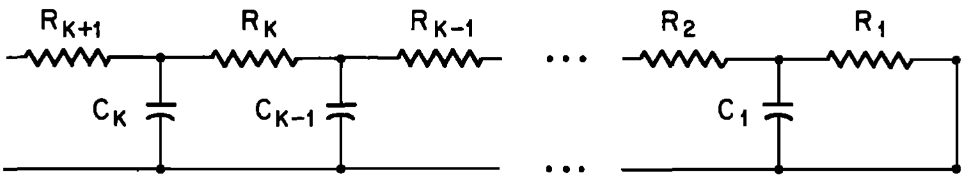

Fig. 1. Equivalent electrical ladder network giving an impedance identical to the admittance of a conductivity model of delta functions.

nections between $\bar{C}$ and P to c and ω it follows that the spectrum of the system is set of values $\lambda = \pi^2(r_k + 2n)^2/d^2$ where n takes on all integer values from $-\infty$ to ∞ and there are K values of r_k in the range -1 to $+1$. We shall see later that the pairs of equations (12), (13) and (14), (15) form the basis of a direct solution to the problem of construction.

The final special class is that of twice-differentiable, strictly positive functions: $\sigma \in C^{2+}$. *Weidelt's* [1972] thorough treatment relieves us of any obligation to go into details. We have already mentioned the special expansion of the admittance (10) and the fact that (1) possesses an infinite spectrum of simple eigenvalues, whose eigenfunctions are complete. The general element of S^+ does not, of course, exhibit these properties, as D^+ demonstrates.

EXISTENCE

Initially we discuss the inverse problem with exact data as in (3). By the representation (9) when a solution in S^+ exists, it is always possible to find a constant $b_0 \geq 0$ and a nondecreasing function b of bounded variation such that

$$c_j = b_0 + \int_0^\infty \frac{1 - i\omega_j\lambda}{\lambda + i\omega_j}\, db(\lambda) \quad j = 1, 2, \cdots N \tag{16}$$

This equation represents $2N$ real linear constraints on an unknown element restricted to lie in a positive cone of an appropriate vector space; the self-consistency of these constraints can be determined by looking at the problem as an infinite-dimensional version of linear programing [*Sabatier,* 1977; *Luenberger,* 1969]. Our treatment will be elementary. Consider the approximation of the Stieljes integral by a real finite sum [*Korevaar,* 1969, p. 408]:

$$b_0 + \sum_{l=1}^{L} G_{ml}\Delta b_l = d_m \quad m = 1, 2, \cdots 2N \tag{17}$$

where

$$d_{2j-1} + i d_{2j} = c_j \quad j = 1, 2, \cdots N$$

the G_{ml} are real and $\Delta b_l = b(\lambda_{l+1}) - b(\lambda_l)$ with $\lambda_{l+1} = \lambda_l + \Delta\lambda$, $\lambda_1 = b(0) = 0$ and $\Delta\lambda$ a positive constant. We consider the limits of L becoming large, $\Delta\lambda$ becoming small in such a way that $L\,\Delta\lambda$ grows without bound. In addition to (17) we have b_0, $\Delta b_l \geq 0$. The fundamental theorem of linear programing [*Luenberger,* 1973] states that when the matrix G_{ml} has rank $2N$, if there is any set of numbers b_0, Δb_l satisfying these constraints, then there must be a set in which at most $2N$ of them are nonzero, the rest vanishing. This holds of course no matter how large L is, that is, no matter how well (17) approximates (16). It suggests that a necessary condition for the existence of a solution $\sigma \in S^+$ is that there must be a function b consisting of at most $2N$ jumps.

The above argument depends upon the rank of G_{ml} being full, and this is hard to verify in general. To get a valid scheme which is also computationally more practical, we modify the problem slightly. First we obtain an integral on a finite interval, for example, with a mapping like $\tilde{\lambda} = (\lambda_0 + \lambda)^{-1}$, where $\lambda_0 > 0$. Next we set up a form of the system for which a solution is guaranteed to exist even when (16) is incompatible with the constraints on b_0 and b. To do this we relax the equality constraint in (16) and (17) and merely seek a solution that minimizes the disagreement in some sense. An easy way to do this is with the system of inequalities

$$-A \leq G_m[b_0, \Delta b_l] - d_m \leq +A \quad m = 1, 2, \cdots 2N \tag{18}$$

where $A \geq 0$ and G_m is an abbreviation for the left side of (17) suitably mapped onto a finite interval. Now we minimize A, which is the sup norm of the misfit. The linear program for this problem is an elementary exercise in the use of slack variables [*Luenberger,* 1973], whose introduction ensures that the matrix of coefficients is not rank deficient. If there is an exact solution (that is, one with $A = 0$), then an application of the fundamental theorem shows that there will be one with no more than $2N$ nonvanishing elements in the variable vector; linear programing algorithms will always find such a solution if one exists. We imagine systematically improving the approximation by finer sampling: Sabatier's analysis shows that A tends (rapidly) to a limit; if that limit is nonzero, this implies that no solution exists to the original problem. Conversely, we shall show in the next section that if A tends to zero and hence a suitable function b and constant b_0 satisfying (16) can be found, we can always construct a corresponding $\sigma \in S^+$; that σ will be in D^+ with no more than $2N$ delta functions. Actually, in practical computations, when a solution exists, it is invariably observed that A becomes exactly zero very quickly (Sabatier's 'saturation'); there is no need to continue refining the approximation, because (17) then represents a particular exact solution to (16).

Another way to characterize the existence result is the following: if there is any solution to the inverse problem in S^+ corresponding to a given data set, then there must be one in D^+. Because the solutions are not normally unique, we expect an infinite variety of conductivities to be associated with a particular set of data, and some of them ought to be ordinary functions rather than distributions. As we shall see, there is a systematic way of finding solutions in H^+, but a question that naturally arises is whether there are data sets connected with a solution in delta functions but not associated with any ordinary function. There are such peculiar data, as we now show with an example. Suppose that c has been exactly determined at two frequencies and that the admittances c_1, c_2 are consistent with a function b that has only one point of discontinuity (which corresponds to a model in D^+ with only one delta function). In these circumstances the model fitting the data is unique. We use distributions here because the Stieljes integral notation is a little cumbersome. Abbreviating the kernels by Γ_j, (16) becomes

$$c_j = b_0 + \int_0^\infty \Gamma_j(\lambda)\, \Delta b_1\, \delta(\lambda - \lambda_1)\, d\lambda \quad j = 1, 2$$

where λ_1 is the point at which b is discontinuous with a positive jump Δb_1. The assumption that there is another pair $\tilde{b}_0$, $\tilde{b}$ satisfying the data leads to a contradiction as follows. Writing $\tilde{b}'$ for the (distributional) derivative of $\tilde{b}$, we have

$$c_j = \tilde{b}_0 + \int_0^\infty \Gamma_j(\lambda)\, \tilde{b}'(\lambda)\, d\lambda \quad j = 1, 2 \tag{19}$$

where $\tilde{b}' \geq 0$. Subtract the equations with $j = 1$ and also subtract those with $j = 2$, then weight the resulting expressions by the complex constants α_1, α_2 and add:

$$0 = (b_0 - \tilde{b}_0)(\alpha_1 + \alpha_2) + \int_0^\infty [\alpha_1\Gamma_1(\lambda) + \alpha_2\Gamma_2(\lambda)]\,[\Delta b_1\, \delta(\lambda - \lambda_1) - \tilde{b}'(\lambda)]\, d\lambda \tag{20}$$

The form of the functions Γ_1, Γ_2 makes it possible to choose α_1, α_2 with these properties: Re $\{\alpha_1 + \alpha_2\} = 0$ and the function $Q = \text{Re}\,\{\alpha_1\Gamma_1 + \alpha_2\Gamma_2\}$ is positive except at $\lambda = \lambda_1$ where it vanishes. Some actual values, which are not particularly illuminating, are $\alpha_1 = 2\lambda_1 + i\omega_1 + \lambda_1^2/i\omega_1$, and $\alpha_2 = -2\lambda_1 - i\omega_2 - \lambda_1^2/i\omega_2$, where $\omega_1 > \omega_2 > 0$. With this choice of constants the real part of (20) becomes

$$0 = \int_0^\infty Q(\lambda)\,\tilde{b}'(\lambda)\,d\lambda$$

Since $Q > 0$ except at $\lambda = \lambda_1$ and $\tilde{b}' \geqslant 0$, this equation is impossible to satisfy together with (19) unless $\tilde{b}'$ has its only support at $\lambda = \lambda_1$, in which case it is easy to see that $\tilde{b} = b$. The argument, which can be generalized to cases with more than two data, is very similar to the one used in ideal body theory [*Parker,* 1975]. The key factor is that the number of constraints (in this example four real numbers) exceeds the number of parameters describing the model (here the three numbers b_0, Δb_1, λ_1). It is conjectured that only when N admittances can be fitted with a model of fewer than N delta functions in conductivity, the solution is unique and, further, that these are the only cases for which there are no ordinary function solutions.

Finally, we consider the theory for data that are not exact. When $c(\omega_j)$ is specified as lying between strict limits (say, inside a rectangle in the complex c plane), the corresponding linear program is easily set up, and the treatment proceeds along the lines already given. If a statistical description of the data uncertainty is prescribed, the question of existence becomes one of a confidence level, for at a sufficiently low probability there will always be a solution associated with any noisy data set. The traditional misfit statistic is χ^2 when the errors are normally distributed (as we shall assume). Now we need to find the model consistent with the constraints that makes χ^2 smallest. Suppose the best-fitting model has an unacceptably low probability of being compatible with the data; then every other model will be worse, and we may reject the hypothesis that there is any model at all. Finding the best-fitting model by varying b is a quadratic programing problem of a well known type [*Lawson and Hanson,* 1974]. It can be shown that when there is no exactly fitting solution (with $\chi^2 = 0$), there is a unique optimal function b consisting of a finite number of discontinuities, and once more $\sigma \in D^+$. This fact explains the observed 'numerical instability' of linearized iterative schemes for the best fitting model: the algorithms gravitate toward wildly oscillatory functions in their attempt to approximate delta functions with smoothly varying ones.

Quadratic programing is not as flexible as the linear kind; for example, it is not easy to apply a quadratic constraint to a problem. Therefore another misfit statistic is sometimes preferable to χ^2 because it requires only the solution of a linear program: the sum of the magnitudes of the misfits normalized by the standard errors [*Gass,* 1975, p. 316]. Confidence and probabilty tables have not been available for this statistic and it has been necessary to use an asymptotic approximation based on the normal distribution [*Banks et al.,* 1977]; recently, however, precise tables have been calculated [*Parker and McNutt,* 1980]. In the rest of the paper we shall treat only the case of exact data unless the appropriate generalization is not straightforward.

CONSTRUCTION

Before taking up the problem of finding the conductivities, we make some remarks about the determination of the layer thickness h. In the global induction problem for the earth, we may assume that even the longest period magnetic variations do not penetrate appreciably into the core; therefore at the base of the mantle there is an effective perfect conductor whose depth is well known. After applying the earth-flattening transformations of Weidelt, a value of h can be supplied. This is simply done because, as (4) and (5) show, $c(0) = h$, and therefore the provision of an additional admittance at zero frequency serves to fix the layer thickness. Similarly, in global induction in other planets without a known core, the earth-flattening mapping takes the center of the planet to the base of the flat layer, with the perfect-conductor boundary condition. For studies at higher frequencies, when the whereabouts of a base is not known ahead of time, the value of h need not be restricted.

Except for certain singular cases of a kind already noted, a finite set of measurements does not define a unique conductivity. Nonetheless, it is often useful to be able to exhibit some of the solutions satisfying given data. First we will describe how this is done for each of the special classes D^+, H^+, C^{2+} defined earlier.

The first class D^+ is the most important because existence of a solution here is a necessary condition for there to be any solution at all. Assume that b_0, b have been found satisfying (16) (or the equivalent constraints corresponding to nonexact data) by the linear programing methods developed in the previous section; the function b will have at most $2N$ points of discontinuity. Therefore using the rearrangement (11), we can put c into the form (13) where $K \leqslant 2N$. The sum may now be rationalized into a ratio of two polynomials in $i\omega$ with real coefficients each of degree K (unless $a_0 = 0$, in which case the numerator will be of degree $K - 1$). By a straightforward manipulation the rational function can be converted into a continued fraction in the form of (12), and so the parameters defining a model in D^+ can be identified directly. This procedure is due to *Krein* [1952]; *Barcilon* [1975] also reviews the essential material in English. As these authors state, the positivity of the determined parameters can be established with a theorem of Stieljes [*Bender and Orszag,* 1978, p. 406].

A continued fraction is also used for finding conductivities in H^+. A value for the electrical thickness parameter $d = (\mu_0\sigma_k h_k^2)^{1/2}$ must be chosen, and the modified admittance $\tilde{c}(\omega_j)$ must be calculated from the values of c_j and ω_j. Now the representation (15) is invoked, but the sum over k is replaced by a Stieljes integral on the real interval $0 \leqslant p \leqslant 1$:

$$\frac{\tilde{c}(\omega_j)}{P_j^2 - 1} = \int_0^1 \frac{2P_j\,dq(p)}{P_j^2 - p^2}$$

The expression has been simplified to account for the symmetry about $p = 0$ and the vanishing of l_1, l_2 mentioned earlier. The existence of a model and some appropriate parameters p_k, q_k can be determined by a linear program which applies the constraint that the function q is nondecreasing. Having obtained constants for the sum (15) (there will be $2N$ or fewer terms in the sum), we rearrange it into a continued fraction based upon (14); *Ghausi and Kelly* [1968, chapter 6] run through a number of explicit examples of this and other similar manipulations. From the continued fraction the value of h_k is determined in each layer and, because d is constant, σ_k is known also. In the case of exact data, solutions will exist for some d and not for others. As the parameter d tends to zero, $\tilde{c} \rightarrow [i\omega d^2]\cdot c$,

$P - 1 \rightarrow i\omega d^2/2$ and (15) approximates (13); thus the models in H^+ come to approximate those in D^+ with thin, highly conductive layers separated by relatively thick, highly resistive layers. We infer from this that if d is small enough there is always a solution in H^+ provided the data are not of the singular kind.

Construction of twice-differentiable solutions is the subject of *Weidelt's* [1972] study. His adaption of *Gel'fand and Levitan's* [1955] classic work shows that if a representation like (13) exists with $a_0 = 0$, $K = \infty$, and constants such that as $n \rightarrow \infty$, $a_n = O(1)$ and $\lambda_n = O(n^2)$, then there are solutions with $\sigma \in C^{2+}$; these can be constructed through Weidelt's procedure. Therefore consider

$$c(\omega) = \int_0^\infty \frac{da(\lambda)}{\lambda + i\omega} + \sum_{k=1}^{\infty} \frac{\epsilon}{k^2 + i\omega}$$

where $\epsilon > 0$. If a has all its variation at a finite number of steps, this is an admittance of the required form. Let us select an arbitrary positive ϵ, put the sum on the left, and evaluate the equation at the frequencies ω_j. We now seek solutions for a with a linear program in such a way as to obtain those with a finite number of discontinuities. A successful solution of this kind for a guarantees a model for σ that is smooth. Again we usually expect solutions if ϵ is chosen small enough, for then the admittance approaches that for a conductivity in D^+.

A lack of uniqueness of solutions persists in the construction methods outlined above. For exact data there will normally be infinitely many valid solutions of the linear programs, and which particular one is discovered (when finding $A = 0$ is the minimum in (18), for example) depends upon unpredictable details of the approximation sequence and linear programing algorithm. Such a lack of definiteness is undesirable. One way to avoid it, once the existence of a solution has been established, is to set up a linear functional like

$$W[b] = w_0 b_0 + \int_0^\infty w(\lambda)\, db(\lambda)$$

to be minimized, where w is a positive continuous bounded real function and w_0 is a real positive constant. When a definite W has been selected, we can be assured the same σ will always be obtained for a specific set of data. Unfortunately, the relationship between W and the solution obtained (it will be in D^+) is unclear because of the complicated nature of the mapping between the function b and the conductivity.

Another approach to the problem of nonuniqueness is that suggested by *Backus and Gilbert* [1967]; a solution is sought nearest in some sense to a preferred model. If the norm on τ in $NBV(0, h)$ is used as a basis for describing distances, finding the nearest element to a fixed element is a complex nonlinear problem, and one can hardly do better than the iterative procedures given by Backus and Gilbert, as most of the literature since their paper attests. However, we have shown that for every $\sigma \in S^+$, there is a pair (b_0, b) in (9) and therefore we may choose to measure separation in the space of these objects:

$$D(\sigma_1, \sigma_2) = ||\mathbf{b}_1 - \mathbf{b}_2||$$

where we define $\mathbf{b} = (b_0, b)$ with $b \in NBV(0, \infty)$ and

$$||\mathbf{b}|| = |b_0| + V_0^\infty[b]$$

The functional D is a topological metric and it requires only a linear program to find the $\mathbf{b}_1$ nearest to a fixed $\mathbf{b}_2$ subject to (19) and the other conditions. One problem remains, however: even when $\mathbf{b}$ is known, we may not know how to construct the corresponding σ or even whether it exists. If the fixed model is restricted to the special classes discussed at length in this paper, it is not difficult to devise a scheme allowing the optimal element in S^+ to be approximated arbitrarily well, at least in the sense of the metric of $\mathbf{b}$. When the data are noisy and a solution nearest to some preferred model is desired, the problem should be set up to find the closest solution subject to the misfit being acceptably low. In this case the misfit criterion is applied as a constraint rather than as the penalty function in the optimization, which makes χ^2 less suitable as a statistic.

SUMMARY

Two mathematical questions have been treated: that of existence of solutions and that of their construction. The first problem has been completely settled, mainly because it can reduced to a linear problem with convex constraints. We have seen that the existence of solutions compatible with incomplete data depends on whether or not delta-function solutions exist, and we have mathematical machinery for deciding this question too. The matter of construction of models fitting the data is more complex because, among other things, the solutions are not normally unique. When solutions have been shown to exist, however, we have the means of constructing models in three special classes of conductivity functions, and these may be assumed to cover most cases of geophysical interest.

In a future paper I hope to explore the practical consequences of the theory set out here, in particular, developing numerically stable procedures and applying them to actual measurements. The most important unanswered question is that of the valid inferences that can be drawn about the actual conductivity from a given collection of data. In a nonlinear problem like this one the most fruitful approach appears to be that of optimizing functionals of σ using the data as constraints. A variational technique naturally suggests itself here, and it is encouraging that the data c_j are in fact Fréchet differentiable for all of S^+. Recent work by *Barcilon* [1979] suggests that when linear functionals of σ are minimized, the class of delta-function elements D^+ may occur as the optimizing solutions; therefore it seems likely that the framework set up for the results of the present paper may be a suitable one for the solution of this very important problem.

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