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TRANSMISSION SYSTEMS AND NETWORK ANALOGIES

TO

GEOPHYSICAL FORWARD AND INVERSE PROBLEMS

TECHNICAL REPORT

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

Transmission system and network analogies to geophysical forward and inverse problems. Applications to one, two, and three dimensional problems. The use of network theorems in developing reciprocity and sensitivity relationships that simplify the calculations for solving the inverse boundary value problem.

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Introduction

In the course of our studies of the electromagnetic environment of the earth, ionosphere, magnetosphere system we have been led several times to numerical solutions of potential or wave propagation problems. The numerical approach was necessary because of the complicated geometry. In all cases, the equations were analogous to transmission systems. These systems have a special form and their difference equation approximations are analogs of networks which also have special simple forms. Since both transmission systems and networks have been extensively analysed, one can apply very useful and powerful concepts, which have been developed for them to the geophysical problems which are their analogs.

All these matters are essentially well known and thus this report does not represent new ideas. The wide applicability of these ideas to geophysical problems, and the fact that this electrical engineering orientation is not the typical orientation of geophysicists leads the author to hope that this report will be of use to other geophysicists.

These ideas were presented to a small group of graduate students in a mini-course during an independent study period this January. For the sake of completeness, a certain amount of background material was also presented, and this material has also been included in this report.

The first chapter reviews the analytic solutions to systems of first order equations. One dimensional field problems fall in this class of equations. A special form of these equations are transmission line equations, and the transformation rules to reduce the equations to this form are developed. Transmission line impedances are defined and the general reflection and transmission laws are shown as well as the Riccatti equation which gives the change of impedance with position.

The second chapter extends the discussion of impedance concepts and shows how these lead to simple approximate treatments of propagation in slowly varying waveguides which are an improvement over the WKB approximations.

The third chapter considers the analogy between transmission system equations and the current-voltage relationships of networks. Since the topology of the network is different than that of the physical space being represented these analogies can be applied to field equations whose vectors do not have the divergenceless and curl free properties of network currents and voltage drops. Examples are shown for electromagnetic, seismic, and potential problems in one, two, and three dimensions. An error analysis of the lumped circuit approximation is also given.

The fourth chapter is a short review of basic network analysis. It also discusses some practical methods of dealing with rectangular networks. Such networks are analogs of two dimensional field problems.

The fifth chapter deals with Telegen's theorem and some of its implications concerning reciprocity. These relationships are applicable to the fields for which the networks were analogs. The use of these reciprocity relationships in simplifying the algorithms for solving the inverse problems of determining the internal physical parameters from boundary measurements is also demonstrated. These simplifications make the inversion of two dimensional problems practical.

The last chapter reviews some basic concepts of matrix analysis and discusses their application to the uniqueness and incompatability problems that arise in the inverse boundary value calculations.

Chapter I. Analytic Solutions of 1st Order Equations

The basic equations that describe the behaviour of physical systems are almost invariably first order equations. The second order equations which are so familiar in classical physics result from eliminating variables of the original set of equations. A certain amount of information is hidden in the second order equations, and since the first order equations are readily adaptable to numerical solutions, there are advantages in staying with the original equations. In Table 1 is shown examples of such systems of equations that are common to geophysical applications

Table 1. Examples of systems of 1st order equations

Potential Fields	$\begin{bmatrix} \nabla\varphi = -F \\ \nabla\cdot F = \rho \end{bmatrix}$
Heat Flow	$\begin{bmatrix} K\nabla T = -Q \\ \nabla\cdot Q = -C\,\partial T/\partial\tau \end{bmatrix}$
Acoustics	$\begin{bmatrix} \nabla P = -\rho\,\partial v/\partial\tau \\ \nabla\cdot(\rho v) = -\rho K\,\partial P/\partial\tau \end{bmatrix}$
E.M.	$\begin{bmatrix} \nabla\times E = -\partial B/\partial\tau \\ \nabla\times H = \sigma E + \epsilon\,\partial E/\partial\tau \end{bmatrix}$

One Dimensional Problems

If the medium parameters do not depend on x, y, or t, one can transform out these dimensions as $\exp(ik_x x + ik_y y - i\omega t)$. Thus the operations of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial t$ are replaced by ik_x, ik_y and $-i\omega$, and the resulting equations only involve derivatives with respect to z. The equations can be written in the form

$$dx/dz = Ax \tag{1.1}$$

As an example we can use the E.M. equations in an isotropic medium

$$\frac{d}{dz}\begin{bmatrix} H_x \\ E_y \\ H_y \\ E_x \end{bmatrix} = \begin{bmatrix} 0 & \sigma' + ik_x^2/\mu\omega & 0 & ik_x k_y/\mu\omega \\ -i\mu\omega + k_y^2/\sigma' & 0 & -k_x k_y/\sigma' & 0 \\ 0 & ik_x k_y/\mu\omega & 0 & -\sigma' - ik_y^2/\mu\omega \\ k_x k_y/\sigma' & 0 & i\mu\omega - k_x^2/\sigma' & 0 \end{bmatrix} \begin{bmatrix} H_x \\ E_y \\ H_y \\ E_x \end{bmatrix}$$

$$H_z = (k_x E_y - k_y E_x)/\mu\omega$$
$$E_z = (ik_x H_y - ik_y H_x)/\sigma'$$
$$\sigma' = \sigma - i\epsilon\omega$$

Matrizant or Propagator Matrix

(Gantmacher, 1960; Frazer, Duncan, and Collar, 1947; Gilbert and Backus, 1966)

A formal solution of (1.1) can be given as (dropping source terms)

$$X(z) = X(z_0) + \int_{z_0}^{z} A(z_1)\, X(z_1)\, dz_1 \tag{1.2}$$

$X(z_1)$ can be expanded in the same way to give

$$X(z) = X(z_0) + \int_{z_0}^{z} A(z_1)\left[X(z_0) + \int_{z_0}^{z_1} A(z_2) X(z_2)\, dz_2\right] dz_1 \tag{1.3}$$

This can be continued by expanding $X(z_2)$ etc. until we have

$$X(z) = \Omega_{z_0}^{z} X(z_0) \tag{1.4}$$

where

$$\Omega_{z_0}^{z} = E + \int_{z_0}^{z} A(z_1)\, dz_1 + \int_{z_0}^{z} A(z_1) \int_{z_0}^{z_1} A(z_2)\, dz_2\, dz_1 + \cdots \tag{1.5}$$

Equation (1.5) converges for all finite valued A. If A is finite then from (1.1) we have that X is continuous and therefore

$$\Omega_{z_0}^{z} = \Omega_{z_1}^{z}\, \Omega_{z_0}^{z_1} \tag{1.6}$$

If sources are present (1.1) becomes

$$dX/dz = AX + S \tag{1.1a}$$

The sources cause a change in X and these changes must be included in (1.4) and give

$$X(z) = \Omega_{z_0}^{z} X(z_0) + \int_{z_0}^{z} \Omega_{z_1}^{z} S(z_1)\, dz_1 \tag{1.7}$$

For homogeneous media A is a constant matrix and (1.4) can be integrated to give

$$\Omega = E + A\Delta z + A^2 \Delta z^2/2! + \cdots$$

$$\text{or} \qquad \Omega = e^{A\Delta z} \tag{1.8}$$

From Sylvester's theorem

$$e^{A\Delta z} = \sum_{i=1}^{n} e^{\lambda_i \Delta z} \prod_{s \neq i} (\lambda_s E - A) \Big/ \prod_{s \neq i} (\lambda_s - \lambda_i) \tag{1.9}$$

where λ_i are the eigenvalues of A.
From (1.9) we see that the eigenvalues of A are related to the propagation constants of the medium. This can also be demonstrated by considering (1.1) and looking for solution of the form e^{ikz}.

From $$dX/dz = AX \tag{1.1}$$

we look for k values such that

$$ikX = AX \tag{1.10}$$

Therefore the allowable ik values are by definition equal to λ ; the eigenvalues of A.

The eigenvectors of A therefore describe the polarization of the characteristic waves of the medium. If two different media which abut each other have A values with similar eigenvectors then a wave from one media can continue into the other media without reflections since the common polarization will satisfy the continuity of X condition.

In some problems critical levels appear where A has a singularity These cause discontinuities in the solutions. For the cases where the singularities are of the form

$$A = B/(z-z_0) \tag{1.11}$$

one can obtain solutions from the integral matrix

$$U = (z-z_0)^B \tag{1.12}$$

since $$dU/dz = BU/(z-z_0) \tag{1.13}$$

Multilayered Media Approximations

A common approximation method used in geophysics is to represent the media as made up of layers, each one of which has constant properties. The properties of each layer are described by the values of the A matrix which are averages of the actual values

$$A_i = (1/\Delta z_i) \int_{z_i}^{z_i+\Delta z_i} A\, dz \tag{1.14}$$

The approximate matrizant is then given from 1.8 as

$$\Omega_{z_1}^{z_n} = e^{A_m \Delta z_m} e^{A_\ell \Delta z_\ell} \cdots e^{A_1 \Delta z_1} \tag{1.15}$$

The Transmission Line Form

Consider a 2x2 system

$$\begin{matrix} V' \\ I' \end{matrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} V \\ I \end{matrix} \quad , \quad \begin{matrix} V' = dV/dz \\ I' = dI/dz \end{matrix} \qquad 1.16)$$

The eigenvalues of A are given as

$$\lambda = \left(Tr \pm \left(Tr^2 - 4(A_{11}A_{22} - A_{12}A_{21})\right)^{1/2}\right)/2 \qquad (.1.17)$$

$$Tr = \text{Trace of } A \quad = A_{11}+A_{22}$$

If the trace is zero, the eigenvalues are equal, but of opposite sign.
The eigenvectors can be represented by their V/I ratios

$$V/I = \left(Tr \pm \left(Tr^2 - 4(A_{11}A_{22} - A_{12}A_{21})\right)^{1/2} - 2A_{22}\right)/2A_{21} \tag{1.18}$$

Even if the trace is zero, the eigenvectors can be different unless the diagonal terms are both zero. This is the form that are called transmission line equations. Setting $A_{12} = -Z$ and $A_{21} = -Y$ we have

$$V' = -ZI$$
$$I' = -YV \qquad (1.19)$$

Z represents an impedance per unit length along the line and Y represents a leakage conductance.
From (1.17)

$$\lambda^2 = ZY = -k^2 \qquad k = \text{propagation constant} \qquad (1.20)$$

From (1.18)

$$V/I = (Z/Y)^{1/2} = K \qquad K = \text{characteristic impedance} \qquad (1.21)$$

The difference equation approximation of (1.19) are voltage and current equations for a network which is called the lumped circuit approximation of the transmission line. Such a circuit is shown in Figure 1. 1.

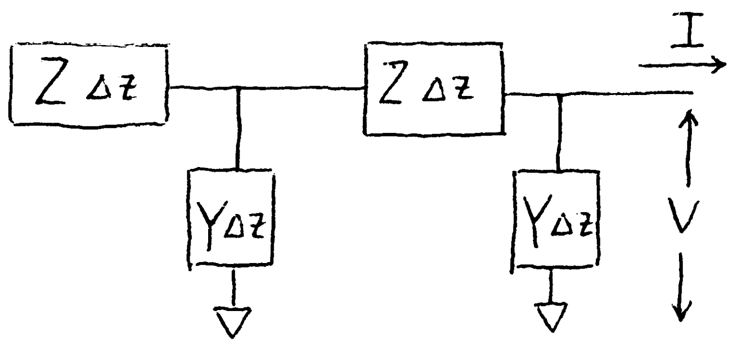


Fig. 1.1 Lumped circuit representation of transmission line

One should notice that the current in the network is a divergenceless vector, while the actual physical field being represented by I may or may not be divergenceless. The transmission line actually brings another dimension into the problem, the direction away from the line to ground, and this allows one to visualize the current as divergenceless. This is a necessary step to making a network analogy, as networks involve divergenceless currents.

Transformations into Transmission Line Equations

If we consider a more general 2nx2n case then A_{11}, A_{12} etc. will be nxn matrices and V and I will be nx1 vectors. In seeking transmission system equations (1.1) can be written as

$$V' - A_{11}V = A_{12}I$$
$$I' - A_{22}I = A_{21}V \qquad (1.22)$$

We seek a transformation

$$\mathcal{V} = B_1 V \qquad \mathcal{I} = B_2 I \tag{1.23}$$

such that

$$V' - A_{11} V = C_1 \mathcal{V} \qquad I' - A_{22} I = C_2 \mathcal{I} \tag{1.24}$$

From (1.23) we can write for (1.24)

$$V' - A_{11} V = C_1 B_1 V' + C_1 B_1' V \qquad I' - A_{22} I = C_2 B_2 I' + C_2 B_2' I \tag{1.25}$$

These equations are satisfied by

$$C_1 = B_1^{-1} \; , \quad B_1' = -B_1 A_{11} \quad \text{or} \quad (B_1^T)' = -A_{11}^T B_1^T$$
$$C_2 = B_2^{-1} \; , \quad B_2' = -B_2 A_{22} \quad \text{or} \quad (B_2^T)' = -A_{22}^T B_2^T \tag{1.26}$$

In order to keep $\mathcal{V}$ and $\mathcal{I}$ continuous when V and I are continuous, B_1 and B_2 must be continuous. The solution for B_1^T and B_2^T is another matrizant problem, but it is of a smaller size because A_{11} and A_{22} are submatrices of A.

The transformed equations become

$$\mathcal{V}' = B_1 A_{12} B_2^{-1} \mathcal{I} \qquad \mathcal{I}' = B_2 A_{21} B_1^{-1} \mathcal{V} \tag{1.27}$$

Impedances and Reflection and Transmission Coefficients

The transmission line form guarantees the same propagation constant for up and down going waves in a homogeneous media and the same polarizations, except for a change of sign in the current variables relative to the voltage variables. This was demonstrated for the 2x2 case, but it also holds for the general form as well. If we seek solutions of (1.19) with $e^{\lambda z}$ dependence we are led to the shifted eigenvalues problem (Lanczos, 1960)

$$[V]\Lambda = -Z[I] \qquad [I]\Lambda = -Y[V] \tag{1.28}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ & & \ddots \end{bmatrix} \qquad \lambda_i = \text{eigenvalues} \tag{1.28a}$$

$$[V] = [V_1 \cdots V_n] \qquad V_i = \text{voltage eigenvector} \tag{1.28b}$$

$$[I] = [I_1 \cdots I_n] \qquad I_i = \text{current eigenvector} \tag{1.28c}$$

From (1.28) we can write

$$[V]\Lambda^2 = -Z[I]\Lambda = ZY[V] \qquad [I]\Lambda^2 = -Y[V]\Lambda = YZ[I] \tag{1.29}$$

$$\therefore \lambda_i^2 \text{ are eigenvalues of ZY or YZ} \tag{1.30}$$

If λ_i is an eigenvalue of (1.28) and $\begin{bmatrix} V_i \\ I_i \end{bmatrix}$ an eigenvector then $-\lambda_i$ is also an eigenvalue and $\begin{bmatrix} V_i \\ -I_i \end{bmatrix}$ an eigenvector as can be proven by direct substitution in (1.28). This proves the contention made above concerning the similarity of up and down going solutions for transmission equations. If we consider $[V]$ and $[I]$ to be the eigenvectors associated with the positive eigenvalues (positive imaginary part) we can define a characteristic impedance as

$$K_0 = [V][I]^{-1} \tag{1.31}$$

K_0 gives us the relationships between V and I for forward propagating solutions. As an example

if $V = [V_1 \cdots V_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then $I = [I_1 \cdots I_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$$K_0 I = [V][I]^{-1}[I][a] = [V][a] = V \tag{1.32}$$

At a boundary between different media one must expect reflections to be set up unless the characteristic impedances of the two media are identical.

$$\begin{aligned} &\text{Let incoming fields be} && V_1^+,\ I_1^+ \\ &\text{reflected fields} && V_1^-,\ I_1^- \\ &\text{transmitted fields} && V_2^+,\ I_2^+ \end{aligned} \tag{1.33}$$

define reflection coefficient matrix as

$$I_1^- = -R I_1^+ \qquad \therefore \quad V_1^- = K_1 R I_1^+ \tag{1.34a}$$

and transmission coefficient matrix as

$$I_2^+ = T I_1^+ \qquad \therefore \quad V_2^+ = K_2 T I_1^+ \tag{1.34b}$$

where $K_1 = (K_0)$ of medium 1, $K_2 = (K_0)$ of medium 2

From the continuity of V we have

$$K_1 + K_1 R = K_2 T \tag{1.35a}$$

and from the continuity of I we have

$$E - R = T, \qquad E = \text{Identity matrix} \tag{1.35b}$$

The solution of (1.35) gives us

$$R = K_1^{-1}(K_2-K_1)\ (K_2+K_1)^{-1}K_1 \tag{1.36}$$

$$T = 2(K_2+K_1)^{-1}\ K_1$$

If the characteristic impedances are identical, R goes to zero and T becomes an identity matrix.

Riccatti's Equation

K_0 was the characteristic impedance of a uniform medium. One can also define an impedance for a non-uniform system, but this impedance will depend on the termination impedance and it will also be a function of position.

Let the termination be associated with a set of eigenvectors $[V]$ and $[I]$ so that $K_T = [V][I]^{-1}$. From the changes in $[V]$ and $[I]$ as we progress away from the termination we can trace out the changes in the impedance found looking back down at the termination

$$dK/dz = \left(d[V]/dz\right)[I]^{-1} + [V]\left(d[I]^{-1}/dz\right) \tag{1.37}$$

since $\quad [I][I]^{-1} = E \quad$ and $\quad dE/dz = 0$

we have $\quad d[I]^{-1}/dz = -[I]^{-1}\left(d[I]/dz\right)[I]^{-1}$

using (1.19) $\quad dK/dz = -Z[I][I]^{-1} + [V][I]^{-1}Y[V][I]^{-1}$

or
$$dK/dz = -Z + KYK \tag{1.38}$$

Equation (1.38) is called Riccatti's equation. It is non-linear but of smaller dimensions than the original set of 1st order equations and can be used effectively for numerical solutions. In many geophysical problems knowing the impedance is all that is required.

Perturbations

In many problems one may have a solution for one particular environment but one wishes to find an approximate solution for a slightly different (perturbed) environment. Let the known solution be X and the environment is represented by A

from $\quad X' = AX \quad$ we have $\quad X(z) = \Omega_{z_0}^{z} X(z_0)$

The desired solution for an environment represented by ($A + \Delta A$) is ($X + \Delta X$)

ignoring products of small quantities

$$\Delta X \simeq \Omega_{z_0}^{z} \Delta X(z_0) + \Delta\Omega_{z_0}^{z} X(z_0) \tag{1.39}$$

also from
$$(X + \Delta X)' = (A + \Delta A)(X + \Delta X)$$

we have
$$\Delta X' \simeq A\Delta X + \Delta A X \tag{1.40}$$

from (1.7)
$$\Delta X \simeq \Omega_{z_0}^{z} \Delta X(z_0) + \int_{z_0}^{z} \Omega_{z_1}^{z} \Delta A\, \Omega_{z_0}^{z_1} X(z_0)\, dz_1 \tag{1.41}$$

∴ comparing (1.34) and (1.41)

$$\Delta\Omega_{z_0}^{z} \simeq \int_{z_0}^{z} \Omega_{z_1}^{z} \Delta A(z_1)\, \Omega_{z_0}^{z_1}\, dz_1 \tag{1.42}$$

Chapter II. Mode Impedances in Slowly Varying Waveguides

The transformation of our equations into a transmission line form was really a process for finding current and voltage like variables in the physical system. These one dimensional concepts are also applicable as approximations to a class of two dimensional problems, which involve slowly varying waveguides.

In finding waveguide solutions one solves a boundary value problem in one dimension, say the Z direction, with certain impedance conditions at the end faces. These conditions can only be satisfied for certain k_x values for any given ω and the solutions at these ω and k_x values are called modes. When the boundary conditions reflect back all the energy the modes have Z variations that have certain orthogonal properties and these modes are called normal modes. In any case, the mode geometries are quite distinctive and independent even when they are not normal modes. If the waveguide should change its properties slowly as a function of X, the direction along the guide, one would expect slow variations in the mode configurations and one should be able to patch solutions together along X without much mode-mode coupling. In such situations one no longer has to keep track of the details of the Z dependence of the solutions, but one can treat each node as entity whose propagation in the X direction can be described by a one dimensional system. For certain simple geometries this can be done exactly, and for many practical problems this can be done approximately, but in every case one must choose the proper averages of the actual field parameters to be the current and voltage variables. Even if mode-mode coupling becomes important one can still use this concept, but in order to compute the coupling coefficients one must look at the details of the Z dependence of the modes.

Perturbation techniques allow us to find the variations in the propagation constant for a mode due to small variations of the media, and one can use this information to make a WKB approximation. One cannot correctly predict the effect of partial reflections, however, until one has defined a proper mode impedance. The transmission line analogy to the mode propagation gives us such an impedance and is therefore in this sense more powerful than the WKB approximation.

As an introduction to these ideas let us consider an acoustic problem in spherical coordinates with no θ or ϕ dependence. The equations for this case are

$$\partial P/\partial r = i\omega\rho v_r \; ; \; \partial v_r/\partial r = -2v_r/r + i\omega K P \tag{2.1}$$

Using 1.26 we can formally transform these equations to a transmission line form with

$$B_1 = 1, \; B_2' = 2B_2/r, \; \therefore \; B_2 = r^2/r_0^2$$

From (1.27) we obtain

$$\partial/\partial r \begin{bmatrix} P \\ r^2 v_r/r_0^2 \end{bmatrix} = \begin{bmatrix} 0 & i\omega\rho r_0^2/r^2 \\ i\omega K r^2/r_0^2 & 0 \end{bmatrix} \begin{bmatrix} P \\ r^2 v_r/r_0^2 \end{bmatrix} \tag{2.2}$$

This represents a transmission line with a propagation constant equal to $\omega(\rho K)^{1/2}$, but a variable characteristic impedance equal to $r_0^2(\rho/K)^{1/2}/r^2$. The variable impedance as a function of r alters the propagation and leads to the spherical bessel function solutions.

One could have written (2.2) more directly by looking for the current like and voltage like variables. In the acoustic problem total flow is the proper current like variable, since this has the property of being divergenceless unless the pressure is varying. This implies a variable proportional to $r^2 v_r$ which is the form we have in (2.2). Pressure is a proper voltage like variable since it will have no gradients in the absence of any accelerations. A further check on the choice of variables can be made by seeing if the product of the current and voltage variables correctly gives the energy flow.

Thus in our acoustic example we would write for the divergence of matter flow

$$d(4\pi r^2 v_r)/dr = 4\pi r^2 i\omega K P \tag{2.3a}$$

and for the gradient of pressure

$$dP/dr = i\omega\rho v_r = i\omega\rho(4\pi r^2 v_r)/4\pi r^2 \tag{2.3b}$$

These equations are equivalent to (2.2)

If one considered acoustic propagation in a cylindrical wedge one would use $r v_r$ as a current variable and P as a voltage variable. The characteristic impedance would then become $(\rho/K)^{1/2}/r$ and this leads to cylindrical bessel function solutions.

As another example consider electromagentic waves propagating in a cylindrical wedge with an E_θ, H_z and no z dependence (Figure 2.1). With perfectly conducting walls, there should be no E_r on the walls. From the integral form of Maxwell's equations

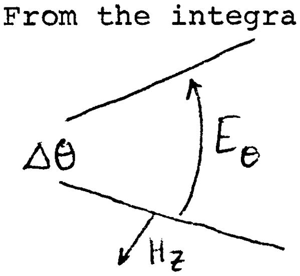


Figure 2.1 Wedge Geometry for EM Waves

and using $E_r = 0$ one can write directly

$$\begin{aligned} d(r\Delta\theta E_\theta)/dr &= i\mu\omega r\Delta\theta H_z \\ dH_z/dr &= i\epsilon\omega E_\theta = (i\epsilon\omega/r\Delta\theta)(r\Delta\theta E_\theta) \end{aligned} \tag{2.4}$$

which is in the transmission line form. In this problem since H_z is proportional to actual radial electric current in the walls of the guide it is natural to take H_z as a current like variable. Since it is $r\Delta\theta E_\theta$ which must be conserved in the absence of time variations of H_z it becomes the voltage variable. The voltage could also have been determined from considering the energy flow which is $r\Delta\theta E_\theta H_z^*$.

These results were arrived at easily because we were using the natural coordinate system for the particular geometry. From these examples, however, we can see how to go about setting up these analogies from the physics of the problem. Since V and I are constant along a transmission unless some I and V is present, the physical parameters chosen to be represented by V and I must have the same property. A further check is provided from the energy flow since VI^* represents the total energy flow. Knowledge of other features of the problem such as the propagation constant can be used to simplify the determination of the transmission line equations.

As an example of a slightly more complicated case, let us consider a higher order mode problem. New components will appear, but we can still use a transmission line analogy. For a simple EM waveguide we will have

$$\begin{aligned} E_z &\sim \cos(k_z z) \\ H_y &\sim \cos(k_z z) \\ E_x &\sim \sin(k_z z) \end{aligned} \qquad (2.5)$$

For a given guide at a given frequency we would determine a value for k_x which turns out to be

$$k_x^2 = k^2 - k_z^2$$

Our equations for studying the x propogation are

$$\partial E_z/\partial x = -i\mu\omega H_y + \partial E_x/\partial z \qquad (2.6a)$$

$$\partial H_y/\partial x = -i\epsilon\omega E_z \qquad (2.6b)$$

From our previous discussion of current and voltage variables we would choose

$$\begin{aligned} I &= H_y \cdot \text{width} \\ V &= E_z \cdot \text{height} \end{aligned} \qquad (2.7)$$

From (2.6b) we have

$$dI/dx = -(i\epsilon\omega \cdot \text{width/height})V \qquad (2.8a)$$

The presence of a $\partial E_z/\partial x$ term in (2.6a) must be accounted for in our transmission line equations. We have already determined k_x however, and since

$$k_x^2 = -ZY \qquad (1.20)$$

and since Y is given in (2.8a) we can write

$$dV/dx = (k_x^2/i\epsilon\omega)(\text{height}/\text{width})I \tag{2.8b}$$

As a check on this result let us consider a wedge waveguide. In this geometry

k_z	becomes	$n\pi/r\theta$
k_x	becomes	$(k^2-(n\pi/r\theta)^2)^{1/2}$
height	becomes	$r\theta$

and width stays constant. Equations (2.8a) and (2.8b) can thus be written

$$V' = ((k^2-\nu^2/r^2)r\theta/i\epsilon\omega)I$$
$$I' = -(i\epsilon\omega/r\theta)V \tag{2.9}$$

where $\nu = n\pi/\theta$

Eliminating V from these equations we have

$$r^2I'' + rI' + (k^2r^2-\nu^2)I = 0 \tag{2.10}$$

The solutions of (2.10) are Bessel functions of order ν which is the correct solution for a wedge guide

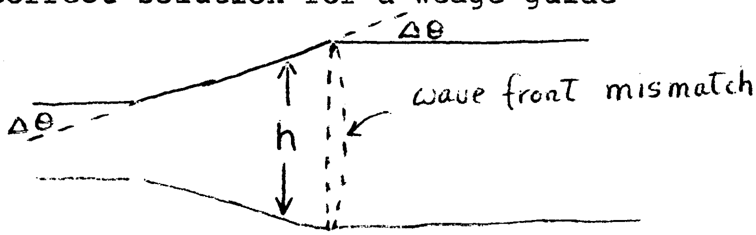


Fig. 2.2 Tapered Waveguide Made up of Wedge Sections

If we connect together wedge sections to make a variable height waveguide we can use equation (2.8). The variable transmission line solutions will not be exact solutions of the tapered waveguide as the wavefront curvatures are not exactly equal at the taper junctions, but as long as $\Delta\theta\, h \ll$ wavelength, the errors will be small. From (2.8) and (1.21) we have as the mode impedance

$$K = (k_x/\epsilon\omega)(\text{height}/\text{width}) \tag{2.11}$$

For zero'th order modes one can have wavelengths much longer than the height of the guide and it is thus possible for the impedance to change appreciably in distances less than a wavelength without violating the conditions for little mode-mode coupling.

Chapter III. General Transmission Systems and Network Approximations

One Dimensional Cases

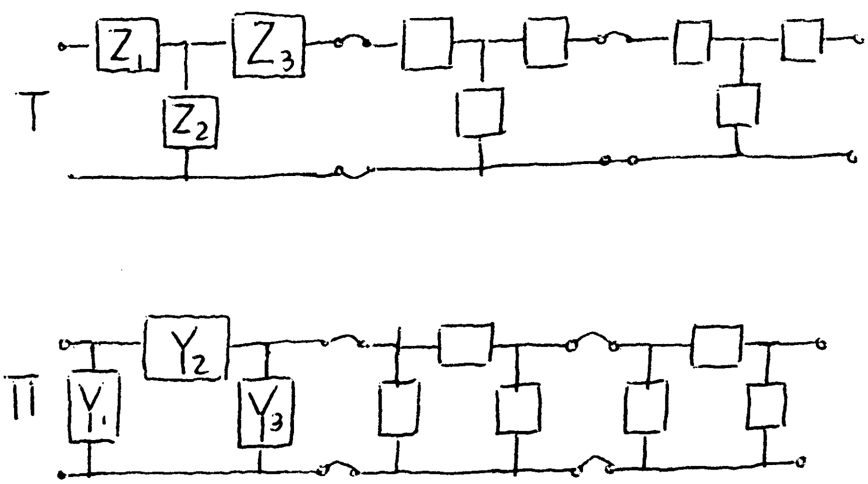
From the transmission line equations

$$\begin{aligned} dV/dZ &= -\ ZI \\ dI/dZ &= -\ YV \end{aligned} \qquad (1.19)$$

one can obtain a simple difference equation approximation

$$\begin{aligned} \Delta V &= -\ ZI \\ \Delta I &= -\ YV \end{aligned} \qquad (3.1)$$

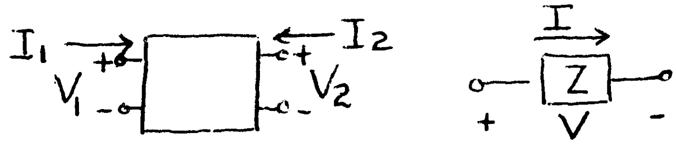
Equation (3.1) can be represented by a ladder network as shown in Figure (1.1). This ladder can be considered the cascading of a string of 4 port networks. Examples of T and Π networks representing (3.1) are shown in Figure 3.1. Z is used to represent impedances and Y represents admittances.



$Z_1 = Z\Delta z/2 = Z_3$
$Z_2 = (Y\Delta z)^{-1}$

$Y_1 = Y\Delta z/2 = Y_3$
$Y_2 = (Z\Delta z)^{-1}$

Figure 3.1 T and Π network representation of transmission lines



$V = ZI\ ,\ I = YV$
$Y = Z^{-1}$

Figure 3.2 Current and voltage conventions for network ports and elements

The network sections can be represented by their impedance and admittance matrices.
For the T section the impedance matrix is defined by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (Z_1+Z_2) & Z_2 \\ Z_2 & (Z_2+Z_3) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad (3.2a)$$

For the Π section the admittance matrix is defined by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} (Y_1+Y_2) & -Y_2 \\ -Y_2 & (Y_2+Y_3) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (3.2b)$$

In the general one dimensional case the current and voltage variables are column vectors of several components and the impedance and admittance elements are matrices. For those cases where the impedance or admittance elements are symmetric matrices they can be represented by an arrangement of two terminal elements as is shown in Figure 3.3

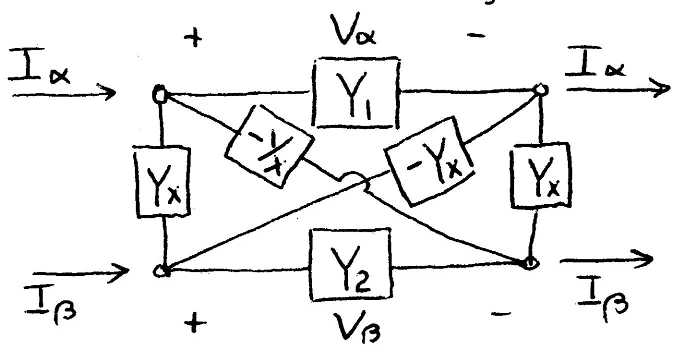


Figure 3.3 Symmetric 2x2 network element

The balance of cross-connections values forces the continuity of I_α and I_β in passing through the element and the element admittance can be written as

$$\begin{bmatrix} I_\alpha \\ I_\beta \end{bmatrix} = \begin{bmatrix} Y_1 & -Y_x \\ -Y_x & Y_2 \end{bmatrix} \begin{matrix} V_\alpha \\ V_\beta \end{matrix} \qquad (3.3)$$

<u>Geophysical Examples of General Transmission Lines</u>

<u>A. Electromagnetic Waves</u>

Consider a plane electromagnetic wave propagating through a medium whose properties are only functions of z. Assuming an $e^{ik_x x + ik_y y - i\omega t}$ dependence and eliminating E_z and H_z, Maxwell's equations take the form

$$\frac{d}{dz}\begin{bmatrix} E_x \\ E_y \\ H_y \\ H_x \end{bmatrix} = \begin{bmatrix} 0 & 0 & i\mu\omega - k_x^2/\sigma' & k_x k_y/\sigma' \\ 0 & 0 & -k_x k_y/\sigma' & -i\mu\omega + k_y^2/\sigma' \\ -\sigma' - ik_y^2/\mu\omega & ik_x k_y/\mu\omega & 0 & 0 \\ -ik_x k_y/\mu\omega & \sigma' + ik_x^2/\mu\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ H_y \\ H_x \end{bmatrix} \qquad (3.4)$$

$$\sigma' = \sigma - i\epsilon\omega$$

If we let

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = V \qquad \text{and} \qquad \begin{bmatrix} H_y \\ -H_x \end{bmatrix} = I \qquad (3.5)$$

then we have $\quad dV/dZ = -ZI$
$\quad\quad\quad\quad\quad\quad dI/dZ = -YV \quad (3.6)$

$$\text{with } Z = \begin{bmatrix} k_x^2/\sigma' - i\mu\omega & k_x k_y/\sigma' \\ k_x k_y/\sigma' & k_y^2/\sigma' - i\mu\omega \end{bmatrix} \tag{3.6a}$$

$$\text{and } Y = \begin{bmatrix} \sigma' + i k_y^2/\mu\omega & -i k_x k_y/\mu\omega \\ -i k_x k_y/\mu\omega & \sigma' + i k_x^2/\mu\omega \end{bmatrix} \tag{3.6b}$$

B. Elastic Waves

Consider P-SV waves in a media varying only in the z direction and an $e^{ik_x x - i\omega t}$ dependence

From
$$-i\omega P_{ij} = \lambda\, \partial v_k/\partial x_k\, \delta_{ij} + \mu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$$
and
$$-i\omega\rho v_i = \partial P_{ij}/\partial x_j \quad , \quad P_{ij} = P_{ji} \tag{3.7}$$

letting
$$\begin{bmatrix} P_{zz} \\ v_x \end{bmatrix} = V \quad \text{and} \quad \begin{bmatrix} v_z \\ P_{zx} \end{bmatrix} = I \tag{3.8}$$

we obtain transmission line equations with
$$Z = \begin{bmatrix} i\omega\rho & ik_x \\ ik_x & i\omega/\mu \end{bmatrix} \quad Y = \begin{bmatrix} i\omega/(\lambda+2\mu) & ik_x\lambda/(\lambda+2\mu) \\ ik_x\lambda/(\lambda+2\mu) & i\omega\rho - i4k_x^2\mu(\lambda+\mu)/(\lambda+2\mu) \end{bmatrix} \tag{3.9}$$

Exact Network Representation of Finite Transmission Line Sections

Equation (3.1) was an approximation to the transmission line equations, and therefore the networks that represented (3.1) are also therefore approximate representations of the original equations. When the transmission line parameters are constant one can use the analytic solutions discussed in Chapter 1 to define an exact impedance or admittance matrix for the entire transmission line section.

Let $[V]$ and $[I]$ represent the voltage and current eigenvector matrices defined in equation (1.28) and let a^+ and a^- be column vectors which represent the magnitudes of each eigenvector represented in the up-going and down-going waves.

At Z = 0 we therefore have

$$V(z = 0) = V_1 = [V]\ (a^+ + a^-)$$
$$I(z = 0) = I_1 = [I]\ (a^+ - a^-) \tag{3.10}$$

and at z = z we have

$$V(z = z) = V_2 = [V](e^{\Lambda z} a^+ + e^{-\Lambda z} a^-)$$
$$I(z = z) = -I_2 = [I](e^{\Lambda z} a^+ - e^{-\Lambda z} a^-) \qquad (3.11)$$

$$\Lambda = \text{eigenvalue matrix} \qquad (1.28a)$$

In (3.11) we have used the sign convention for currents at a port in defining I_2. Combining (3.10) and (3.11) we can write

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} [I] & 0 \\ 0 & [I] \end{bmatrix} \begin{bmatrix} E & -E \\ -e^{\Lambda z} & e^{-\Lambda z} \end{bmatrix} \begin{bmatrix} a^+ \\ a^- \end{bmatrix} \qquad (3.12a)$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} [V] & 0 \\ 0 & [V] \end{bmatrix} \begin{bmatrix} E & E \\ e^{\Lambda z} & e^{-\Lambda z} \end{bmatrix} \begin{bmatrix} a^+ \\ a^- \end{bmatrix} \qquad (3.12b)$$

Solving (3.12a) for a^+ and a^- and using this in (3.12b) gives

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} [V] & 0 \\ 0 & [V] \end{bmatrix} \begin{bmatrix} -i\tan^{-1}(i\Lambda z) & -i\sin^{-1}(i\Lambda z) \\ -i\sin^{-1}(i\Lambda z) & -i\tan^{-1}(i\Lambda z) \end{bmatrix} \begin{bmatrix} [I]^{-1} & 0 \\ 0 & [I]^{-1} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad (3.13a)$$

or

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} [I] & 0 \\ 0 & [I] \end{bmatrix} \begin{bmatrix} -i\tan^{-1}(i\Lambda z) & +i\sin^{-1}(i\Lambda z) \\ +i\sin^{-1}(i\Lambda z) & -i\tan^{-1}(i\Lambda z) \end{bmatrix} \begin{bmatrix} [V]^{-1} & 0 \\ 0 & [V]^{-1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (3.13b)$$

(3.13a) and (3.13b) define the network impedance matrix and the network admittance matrix respectively.

Comparing equations (3.13) and (3.2) one can identify the network section elements and using the identity

$$\sin^{-1}(z) - \tan^{-1}(z) = \tan(z/2) \qquad (3.14)$$

we have for a T network representation

$$Z_2 = -[V]\,[i\sin^{-1}(i\Lambda z)]\,[I]^{-1}$$
$$Z_1 = Z_3 = +[V]\,[i\tan(i\Lambda z/2)]\,[I]^{-1} \qquad (3.15)$$

or for a Π network representation

$$Y_2 = -[I]\ [i \sin^{-1}(i\Lambda z)]\ [V]^{-1}$$
$$Y_1 = Y_3 = +[I]\ [i \tan(i\Lambda z/2)]\ [V]^{-1} \qquad (3.16)$$

For thin layers when $\Lambda z = \Lambda \Delta z \ll 1$

$$\sin(i\Lambda\Delta z) \cong \tan(i\Lambda\Delta z) \cong i\Lambda\Delta z$$

and from (1.28) we have

$$\Lambda^{-1}[V]^{-1} = -\ [I]^{-1}\ Z^{-1} \qquad (3.17a)$$

$$\Lambda^{-1}[I]^{-1} = -\ [V]^{-1}\ Y^{-1} \qquad (3.17b)$$

Combining (3.15), (3.17b) and (1.28) we have for T network

$$Z_2 \cong Y^{-1}/\Delta z$$
$$Z_1 = Z_3 \cong Z\ \Delta z/2 \qquad (3.18)$$

Combining (3.16), (3.17a), and (1.28) we have for Π network

$$Y_2 \cong Z^{-1}/\Delta z$$
$$Y_1 = Y_3 \cong Y\ \Delta z/2 \qquad (3.19)$$

(3.18) and (3.19) are equivalent to the difference equation approximation (3.1).

When we wish to cascade sections it is best to define a transmission matrix for the network section. This is the same as the matrizant or propagator matrix discussed in Chapter I, but it can also be derived directly from the impedance or admittance matrix.

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} [V] & 0 \\ 0 & [I] \end{bmatrix} \begin{bmatrix} \cos(i\Lambda z) & -i\sin(i\Lambda z) \\ -i\sin(i\Lambda z) & \cos(i\Lambda z) \end{bmatrix} \begin{bmatrix} [V]^{-1} & 0 \\ 0 & [I^{-1}] \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \qquad (3.20)$$

The relationships between the impedance, admittance, and transmission matrices are given below

$$[Z] = \begin{bmatrix} -T_{21}^{-1}T_{22} & -T_{21}^{-1} \\ (T_{12} - T_{11}T_{21}^{-1}T_{22}) & -T_{11}T_{21}^{-1} \end{bmatrix} = [Y]^{-1} \tag{3.21a}$$

$$[Y] = \begin{bmatrix} -T_{12}^{-1}T_{11} & T_{12}^{-1} \\ (T_{22}T_{12}^{-1}T_{11} - T_{21}) & -T_{22}T_{12}^{-1} \end{bmatrix} = [Z]^{-1} \tag{3.21b}$$

$$[T] = \begin{bmatrix} Z_{22}Z_{12}^{-1} & Z_{21} - Z_{22}Z_{12}^{-1}Z_{11} \\ -Z_{12}^{-1} & Z_{12}^{-1}Z_{11} \end{bmatrix} = \begin{bmatrix} -Y_{12}^{-1}Y_{11} & Y_{12}^{-1} \\ (Y_{22}Y_{12}^{-1}Y_{11} - Y_{21}) & -Y_{22}Y_{12}^{-1} \end{bmatrix} \tag{3.21c}$$

One can compute the transmission matrix of the cascaded sections by matrix multiplication of the individual section transmission matrices. The overall transmission matrix can then be converted into an impedance or admittance matrix, but it is not necessarily clear that these matrices will have the symmetry of a simple T or π network. Later it will be shown, as an application of Telegen's theorem, that if the individual network elements are symmetric then the network impedance or admittance matrix must also be symmetric. Thus any finite transmission line segment can be represented by a single T or π segment.

Two Dimensional Examples of General Transmission Surfaces

The concepts discussed concerning transmission lines and their network analogs can be extended to higher dimensions and it is in these extensions that the ideas developed here have their greatest usefulness. In making these extensions one must make a more definite distinction between current variables and voltage variables, as the vector properties of the current variables and the scalar properties of the voltage variables become apparent in a way that did not appear in the one dimensional case. In fact an example of a transmission line analog to a seismic problem was given where stress tensor components and velocity components were mixed together. This situtation, described in equations (3.8) and (3.9), came about by accident when the y dependence was taken out.

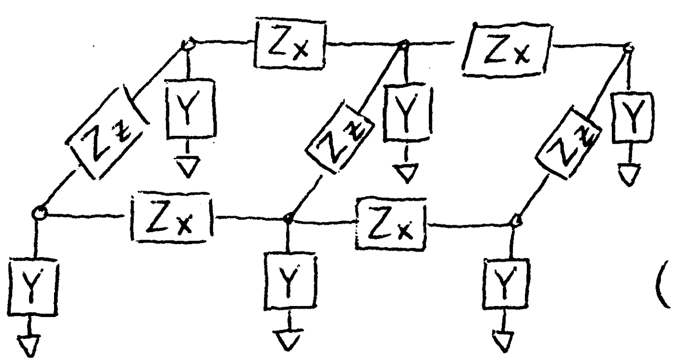
The general form of tranmission surface or volume equations can be given as

$$\nabla V = -Z\underline{I}$$
$$\nabla\cdot\underline{I} = -YV \qquad (3.22a)$$

or equivalently

$$\underline{I} = -Z^{-1}\nabla V$$
$$\nabla\cdot\underline{I} = -Y\,V \qquad (3.22b)$$

In equation (3.22) one has gradient operators acting on the voltage variables, and divergence operators acting on the current variables. An example of a difference equation network analog is shown in Figure 3.4. Again it must be stressed that the network currents are divergenceless, while the current variables in the transmission equations may have a divergence. The leakage path through Y introduces another dimension for



$$Z_x = Z\,\Delta x/\Delta z \qquad Y_x = Z_x^{-1}$$
$$Z_z = Z\,\Delta z/\Delta x \qquad Y_z = Z_z^{-1}$$
$$Y = Y\Delta x\Delta z$$
$$(\text{network current})_{\substack{x\\z}} = \begin{matrix} I_x\,\Delta z \\ I_z\,\Delta x \end{matrix}$$

Figure 3.4 Transmission surface network analog

the network which allows this difference. The voltage and current variables may of course be arrays so that the impedance and admittance elements are in general matrix elements. These can still be represented by simple two terminal network elements as shown in Figure 3.3 whenever the matrices are symmetric. The higher dimensionality of the transmission surface allows the possibility of anisotropic matrix elements, so that potential gradients in one direction can cause current flows in another direction. When the anisotropic terms are symmetric they can also be represented by balanced cross-connections as depicted in Figure 3.3.

Many field problems have very simple representations as transmission surfaces. As an example consider current flow in a medium with no conductivity variations in the y direction. The basic equations are

$$\nabla\varphi = -\rho J \qquad \nabla\cdot J = 0 \tag{3.23}$$

For $\rho = \rho(x,z)$ and $\varphi, J \sim e^{ik_y y}$ we have

$$\begin{aligned} \partial\varphi/\partial x &= -\rho J_x \\ \partial\varphi/\partial z &= -\rho J_z \\ \partial J_x/\partial x + \partial J_z/\partial z &= -(k_y^2/\rho)\,\varphi \end{aligned} \tag{3.24}$$

Figure 3.4 is a network representation of equation (3.24) with

$$\begin{aligned} Z_x &= \rho\,\Delta x/\Delta z \\ Z_z &= \rho\,\Delta z/\Delta x \\ Y &= (k_y^2/\rho)\,\Delta x\,\Delta z \end{aligned} \tag{3.25}$$

Similar results can be obtained for heat flow or acoustic problems.

The seismic equations (3.7) represent a slightly more complicated example, but they are of the form of equations (3.22b). The stress tensor components are current component variables and the velocity components are voltage variables. For the simpler case of no y dependence and no motions in the y direction we have

$$\begin{bmatrix} P_{xx} \\ P_{zx} \end{bmatrix} = -1/(i\omega)\begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}\partial/\partial x\begin{bmatrix} v_x \\ v_z \end{bmatrix} - 1/(i\omega)\begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}\partial/\partial z\begin{bmatrix} v_x \\ v_z \end{bmatrix} \tag{3.26a}$$

$$\begin{bmatrix} P_{xz} \\ P_{zz} \end{bmatrix} = -1/(i\omega)\begin{bmatrix} 0 & \mu \\ \lambda & 0 \end{bmatrix}\partial/\partial x\begin{bmatrix} v_x \\ v_z \end{bmatrix} - 1/(i\omega)\begin{bmatrix} \mu & 0 \\ 0 & \lambda+2\mu \end{bmatrix}\partial/\partial z\begin{bmatrix} v_x \\ v_z \end{bmatrix} \tag{3.26b}$$

$$\partial/\partial x\begin{bmatrix} P_{xx} \\ P_{zx} \end{bmatrix} + \partial/\partial z\begin{bmatrix} P_{xz} \\ P_{zz} \end{bmatrix} = -i\omega\rho\begin{bmatrix} v_x \\ v_z \end{bmatrix} \tag{3.26c}$$

P_{xx} and P_{zx} represent x directed current variables and P_{xz} and P_{zz} represent z directed current variables while U_x and U_z are voltage variables. P_{xx}, P_{xz}, and U_x form one transmission surface system and P_{zx}, P_{zz}, and U_z form another, but they are coupled together by symmetric terms. Figure 3.5 shows the cross-coupling network elements.

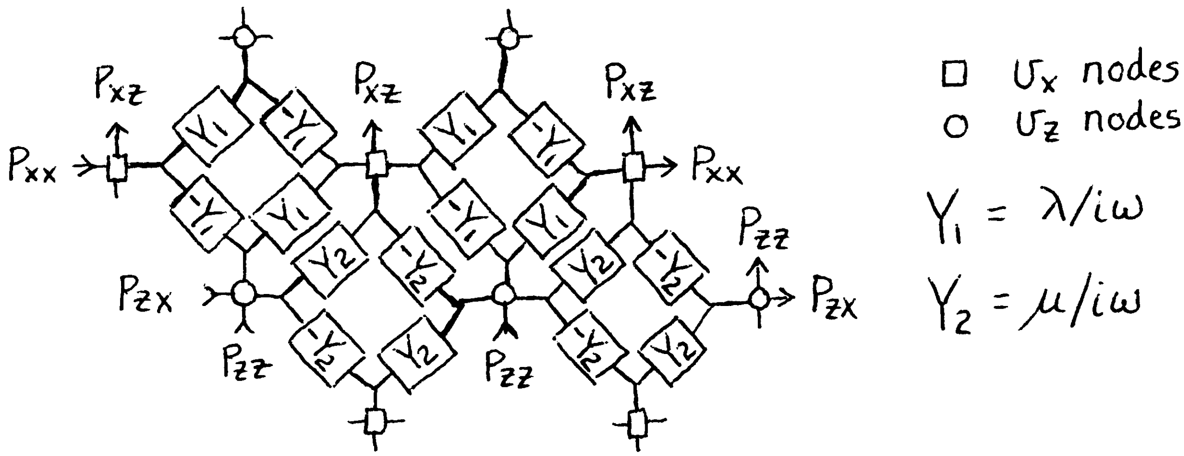


Figure 3.5 Cross-coupling network elements for seismic representation

The rest of the network appears like the network shown in Figure 3.4 with the following values

Table 3.1
Transmission Surface Network Elements for Seismic Representation

U_x network	U_z network
$Y_x = ((\lambda+2\mu)/i\omega)\,\Delta z/\Delta x$	$Y_x = (\mu/i\omega)\,\Delta z/\Delta x$
$Y_z = (\mu/i\omega)\,\Delta x/\Delta z$	$Y_z = ((\lambda+2\mu)/i\omega)\,\Delta x/\Delta z$
$Y = i\omega\rho\,\Delta x\,\Delta z$	$Y = i\omega\rho\,\Delta x\,\Delta z$

Two Dimensional Curl Operations

The electromagnetic equations represent a quite different case, as curl operation are involved and it is not clear that they lead to transmission equations. When the equations are confined to two dimensions, however, one can re-define field variables so that a curl operation becomes a divergence operation on the new variables, and thus transform Maxwell's equations into transmission equations.

Maxwell's equations in two dimensions with an $e^{ik_y y}$ field dependence and no σ variation in the third dimension can be written as

$$
\begin{aligned}
ik_y E_z - \partial E_y/\partial z &= i\mu\omega H_x \qquad & ik_y H_z - \partial H_y/\partial z &= \sigma' E_x \\
\partial E_x/\partial z - \partial E_z/\partial x &= i\mu\omega H_y \qquad & \partial H_x/\partial z - \partial H_z/\partial x &= \sigma' E_y \\
\partial E_y/\partial x - ik_y E_x &= i\mu\omega H_z \qquad & \partial H_y/\partial x - ik_y H_x &= \sigma' E_z
\end{aligned}
\tag{3.27}
$$

Defining voltage and current variables

$$
V = \begin{bmatrix} H_y \\ E_y \end{bmatrix}, \quad J_x = \begin{bmatrix} -E_z \\ H_z \end{bmatrix}, \quad J_z = \begin{bmatrix} E_x \\ -H_x \end{bmatrix} \tag{3.28}
$$

we have

$$
\partial V/\partial x = -\begin{bmatrix} \sigma' & 0 \\ 0 & -i\mu\omega \end{bmatrix} J_x - \begin{bmatrix} 0 & +ik_y \\ -ik_y & 0 \end{bmatrix} J_z \tag{3.29a}
$$

$$
\partial V/\partial z = -\begin{bmatrix} 0 & -ik_y \\ +ik_y & 0 \end{bmatrix} J_x - \begin{bmatrix} \sigma' & 0 \\ 0 & -i\mu\omega \end{bmatrix} J_z \tag{3.29b}
$$

$$
\partial J_x/\partial x + \partial J_z/\partial z = -\begin{bmatrix} -i\mu\omega & 0 \\ 0 & \sigma' \end{bmatrix} V \tag{3.30}
$$

(3.29) can be re-written in terms of admittances as

$$
J_x = -\left(1/(k^2-k_y^2)\right)\left\{\begin{bmatrix} i\mu\omega & 0 \\ 0 & -\sigma' \end{bmatrix}\partial V/\partial x + \begin{bmatrix} 0 & ik_y \\ -ik_y & 0 \end{bmatrix}\partial V/\partial z\right\} \tag{3.31a}
$$

$$
J_z = -\left(1/(k^2-k_y^2)\right)\left\{\begin{bmatrix} 0 & -ik_y \\ ik_y & 0 \end{bmatrix}\partial V/\partial x + \begin{bmatrix} i\mu\omega & 0 \\ 0 & -\sigma' \end{bmatrix}\partial V/\partial z\right\} \tag{3.31b}
$$

The cross coupling network elements for these electromagnetic equations are shown in Figure 3.6. The remaining network elements as shown in Figure 3.4 are listed below in Table 3.2.

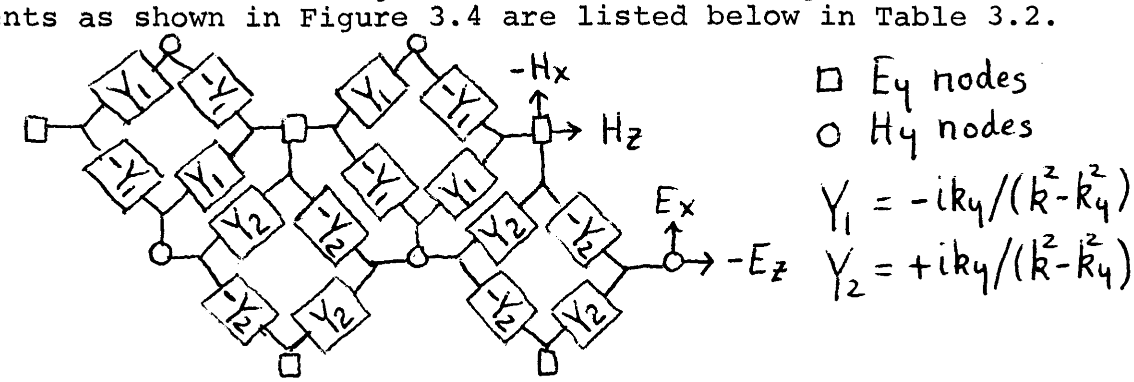


Figure 3.6 Cross coupling network elements for E.M. representation

Table 3.2

Transmission Surface Network Elements for Electromagnetic Fields

H_y network	E_y network
$Y_x = \left[i\mu\omega/(k^2-k_y^2)\right]\Delta z/\Delta x$	$Y_x = \left[-\sigma'/(k^2-k_y^2)\right]\Delta z/\Delta x$
$Y_z = \left[i\mu\omega/(k^2-k_y^2)\right]\Delta x/\Delta z$	$Y_z = \left[-\sigma'/(k^2-k_y^2)\right]\Delta x/\Delta z$
$Y = -i\mu\omega\,\Delta x\Delta z$	$Y = \sigma'\Delta x\Delta z$

When k_y equals zero, the cross coupling admittances also become zero and the two sets of polarizations are decoupled. This is the usual approximation made in low frequency problems such as magnetotellurics. What the network also shows us is that the two polarizations (TE and TM modes) are decoupled in a homogeneous medium even when $k_y \neq 0$. This comes about because of the balance between the cross coupling elements. Figure 3.6 shows that a Y_1 elements is always in parallel with a Y_2 element, and since $Y_1 = -Y_2$ their effects cancel out and decouples the two networks.

At a boundary,however, one element will represent one medium, and the parallel element a different medium so that $Y_1 \neq Y_2$ and coupling occurs. These results are not as easily seen in the original equations since the cancelling currents are actually different electric and magnetic field components. It is an example of the insights that one can gain from the organization of a network representation from the original field equations. Further insights can be gained from the network theorems that apply to these representations, some of which will be discussed in Chapter V.

Three Dimensional Transmission Volume Examples

In most cases computer time and memory limitations make full three dimensional modelling impractical, but the network analogies are still useful conceptually. The vector equations of transmission systems given in (3.22) are applicable to three dimensional systems. Potential problems, diffusion problems, and acoustic problems are directly represented as transmission systems as can be seen from the equations in Table 1.1.

The seismic equations given in equation (3.7) are also represented as transmission systems but cross terms appear. The transmission current variables are

$$\begin{bmatrix} P_{xx} \\ P_{yx} \\ P_{zx} \end{bmatrix} = J_x \qquad \begin{bmatrix} P_{xy} \\ P_{yy} \\ P_{zy} \end{bmatrix} = J_y \qquad \begin{bmatrix} P_{xz} \\ P_{yz} \\ P_{zz} \end{bmatrix} = J_z \tag{3.32a}$$

and the voltage variables are

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = V \qquad (3.32b)$$

From the momentum equation we have uncoupled admittance to ground proportional to $i\omega\rho$. The stress equations show symmetric cross coupling between the three transmission systems. These equations are simply extensions of the two dimensional case given in equation (3.26).

Our analogy breaks down for the three dimensional electromagnetic equations. One cannot organize the equations into divergence like and gradient like operators. In a provocative article (Branin, 1966) Branin has shown that a higher dimensional extension of network theory is applicable, but the usefulness of this approach has not been explored.

Error Analysis of Network Approximation

In order to compare the behavior of a transmission line with that of its network approximate we must compare both the impedances and the propagation behavior. The impedance of the transmission line, K_0 is given as $(Z/Y)^{1/2}$. For an infinite network, since the impedance at one node will act as the terminating impedance for the previous section we have for a T network

$$K = \frac{Z\Delta z}{2} + \cfrac{1}{Y\Delta z + \cfrac{1}{\frac{Z\Delta z}{2} + K}} \qquad (3.33)$$

This reduces to $K^2 = Z^2\Delta z/4 + Z/Y$

or $$K \cong \pm (Z/Y)^{1/2}\left(1 + ZY\Delta z^2/4\right)^{1/2} \qquad (3.34)$$

Thus the fractional error in K is given approximately as

$$\Delta K/K \cong -k^2\Delta z^2/8 \qquad (3.35)$$

The change in current and voltage as we move through the network can also be compared with the transmission line.

I at node n+1 = I at node n $\left(1 \pm (ZY)^{1/2}\Delta z + ZY\Delta z^2/2 \mp (ZY)^{3/2}\Delta z^3/8\right)$ (3.36)

In a transmission line

$$\text{I at } z+\Delta z \;=\; \text{I at } z\left(1 \pm ik\Delta z - k^2\Delta z^2/2 \mp ik^3\Delta z^3/6 + \cdots\right) \qquad (3.37)$$

Thus the fractional error in one node spacing = $k^3\Delta z^3/24$ (3.38)

After n nodes or a distance $L = n\,\Delta z$

$$\text{fractional error in I} \doteq (kL/3)(k^2\Delta z^2/8) \qquad (3.39)$$

The same results apply to Π networks.

For the two dimensional networks the k value that should be used in 3.35 and 3.39 is the k component in the direction of the spacing. Thus for a problem where the propagation is mostly vertical one can use much larger horizontal spacings than vertical spacings. Near inhomogeneities where reflected and refracted fields are set up one has to tighten up the horizontal spacing as well.

Chapter IV. Network Analysis

Topology

Kirchoff node current equations

sum of currents into any node = 0

$$\sum_j a_{kj}\, i_j = 0 \tag{4.1}$$

$a_{kj} = +1$ if i_j leaving k node

$a_{kj} = -1$ if i_j into k node

$a_{kj} = 0$ if i_j does not connect with k

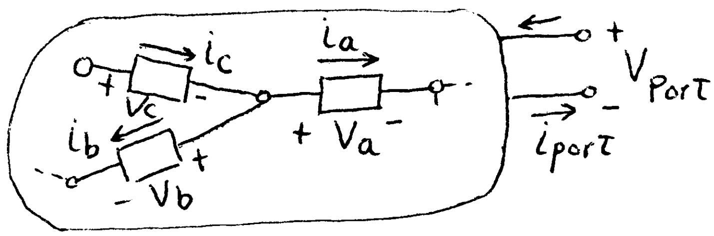


Figure 4.1 Current and voltage drop conventions

a_{kj} called <u>branch-node incidence</u> matrix

n nodes, q branches
n rows of A are not independent since $\sum_k a_{kj} = 0$ (4.2)

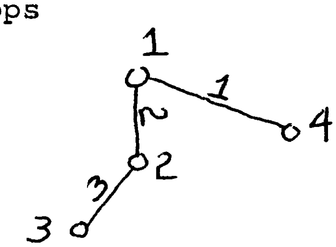
as each branch is incident on 2 nodes
one with + 1, the other with - 1 a_{ij} value

$$a_{ij} \text{ row} = - \sum_{k \neq i} a_{kj} \tag{4.3}$$

There are however (n-1) independent rows or rank of a_{ij} is n-1.

Proof of rank of a_{ij}

Form a tree by connecting n nodes with n-1 branches that form no loops



Canonical numbering of tree
1st branch from node 1 to node n
2nd branch from node 1 to node 2

isolated nodes treated last

3rd branch from node (1 or 2) to node 3
4th branch from node (1 or 2 or 3) to node 4 etc.

Remainder of branches numbered arbitrarily
a_{ij} must now have the form.

$$\left[\begin{array}{cccc|c} \pm 1 & & A_b & & A_\epsilon \\ 0 & \pm 1 & & & \\ 0 & 0 & \pm 1 & & \\ \vdots & \vdots & 0 & \ddots \ \pm 1 & \\ \hline \pm 1 & & & & \end{array}\right]$$

Every row of A_b is obviously independent, therefore (n-1) independent rows in A.
(4.1) can now be written $A_b I_b = -A_\epsilon I_\epsilon$ (excluding nth equation)

$$\therefore \quad I_b = -A_b^{-1} A_\epsilon I_\epsilon \quad \text{since } A_b \text{ non-singular} \tag{4.4}$$

$\therefore$ if q branches, q-(n-1) branch currents, I_ϵ, can be used to determine all currents.

<u>Mesh Currents</u>

Consider our tree, each remaining q-(n-1) branches will form a loop. Define loop or mesh currents each of which traverses one and only one of independent ϵ branches.

Each loop uniquely defined by leaving remaining ϵ branches open, and each loop independent since at least the ϵ part of loop devoid of any other loop currents.

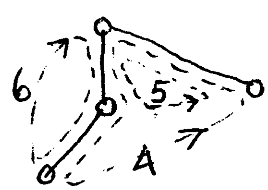


Figure 4.3. Mesh currents around tree.

Each branch current = sum of mesh currents across branch

$$I_b = B' I_L$$
$$I_\epsilon = E I_L \qquad E = \text{identity matrix} \tag{4.5}$$

$B = \begin{bmatrix} B' \\ E \end{bmatrix}$ branch-mesh incident matrix
+1 if branch and mesh are common and in same direction
-1 if branch and mesh are common and in opposite direction
0 if branch and mesh are not common

$$\text{Since } I_L = I_\epsilon, \; I_b = B' I_\epsilon \text{ and } -A_b^{-1} A_\epsilon = B' \tag{4.6}$$

Kirchoff Loop Voltage Equations

$\sum$ voltage drops around loop = 0
Using same orientation for voltage drop as for current we can write

$$\sum B_{ij} V_i = 0 \quad \text{or} \quad B^T V = 0 \tag{4.7}$$

$B^T = [B'^T \; E]$ has rank $\epsilon = q-(n-1)$ since E has ϵ independent rows.

From (4.7)

$$B'^T V_b + E V_\epsilon = 0 \tag{4.8}$$

$$\therefore V_\epsilon = -B'^T V_b$$

or
we have (n-1) independent voltage drops and 4.8 is the dual or transpose relationship of (4.4) or (4.6).

Ohm's Law

Up to this point our network relationships have been purely topological ones and do not involve any details concerning the actual values of voltage or current. To bridge the gap between the topological relations and the actual voltage and current values one must add the laws concerning the behavior of the network elements

$$V = ZI \quad \text{or} \quad I = YV \tag{4.9}$$

When transformers are absent (or gyrators) Z or Y are diagonal matrices.

Thus we can write

$$\begin{bmatrix} V_b \\ V_\epsilon \end{bmatrix} = \begin{bmatrix} Z_b & 0 \\ 0 & Z_\epsilon \end{bmatrix} \begin{bmatrix} I_b \\ I_\epsilon \end{bmatrix} , \quad \begin{bmatrix} I_b \\ I_\epsilon \end{bmatrix} = \begin{bmatrix} Y_b & 0 \\ 0 & Y_\epsilon \end{bmatrix} \begin{bmatrix} V_b \\ V_\epsilon \end{bmatrix} \tag{4.10}$$

Using 4.6, 4.8 and 4.10

$$V_b = Z_b B' I_\epsilon \tag{4.11a}$$

$$I_\epsilon = -Y_\epsilon B'^T V_b \tag{4.11b}$$

Transmission System Admittance Matrices

Transmission system networks have a specially simple topology. Ground represents a common node which connects to each of the other node points. Thus we can construct a tree from all the branches that connect ground to the other node points. In this system the independent mesh currents are then the current variables of the transmission system, and the independent branch voltages are the voltage variables of the transmission system. The ground node is numbered as n in this system. A common boundary value problem will consist of assigning voltage or current values at the boundary. This can be simulated by current sources into the boundary nodes.

If current sources J_b are added and if the ground node is the nth node, the current equation (4.1) can be wirtten as

$$A_b I_b + A_\epsilon I_\epsilon = J_b \tag{4.1a}$$

From 4.10 and 4.11 this can be written as

$$[A_b Y_b - A_\epsilon Y_\epsilon B'^T] V_b = J_b \tag{4.12a}$$

If a voltage source φ_B (from node to ground) is added this can be simulated by a current source

$$J = Y_B \varphi_B - I_B$$

giving

$$[(A_b + E) Y_b - A_\epsilon Y_\epsilon B'^T] V_b = Y_B \varphi_B \tag{4.12b}$$

Current sources across nodes simply become equal but opposite currents into the two nodes. Voltage sources, φ_ϵ , between nodes require a modification of equation (4.10) as

$$V_\epsilon = Z_\epsilon I_\epsilon + \varphi_\epsilon$$

$$I_\epsilon = Y_\epsilon (V_\epsilon - \varphi_\epsilon) = - Y_\epsilon B'^T V_b - Y_\epsilon \varphi_\epsilon$$

giving

$$[A_b Y_b - A_\epsilon Y_\epsilon B'^T] V_b = -A_\epsilon Y_\epsilon \varphi_\epsilon \tag{4.12c}$$

Two Dimensional Network Solutions

The network admittance equations (4.12) have the form

$$YV = J \tag{4.12d}$$

The simple connectivity of a network allows one to determine Y by inspection without going through the formal step of defining branch-node and branch-mesh incident matrices. One simply has to remember that (4.12d) is expressing the continuity of current. This gives the following rule for the elements of the admittance matrix Y.

Diagonal elements Y_{ii} = $\sum$ all admittances connected to node i including admittance from node i to ground

Off-diagonal elements Y_{ij} = - admittance value connecting node i to j.

When voltage sources, φ_B, are included (4.12b), the ground admittance of the nodes involved is doubled.

For rectangular networks one can group the nodes together into rows or columns. This allows one to consider the network as a transmission line. This concept is especially useful when all the measurements and sources are on one edge of the network.

The transmission matrix across a row or column can be quickly arrived at. Consider Figure 4.4 where a column of T sections are grouped together

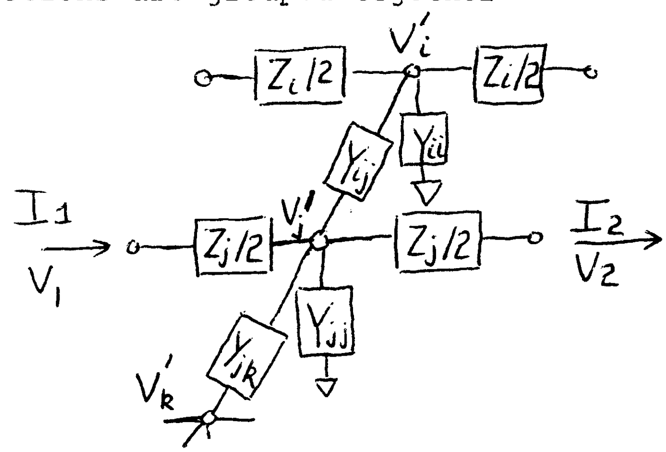


Figure 4.4 Column section of two-dimensional network

Let AV' represent the current lost from I_1 due to flow away from the center through conductances Y_{jj}, Y_{ij}, and Y_{jk}.

Thus $$A = \begin{bmatrix} (Y_{11} + Y_{12}) & -Y_{12} & 0 & 0 & \cdots \\ -Y_{12} & (Y_{12} + Y_{22} + Y_{23}) & -Y_{23} & 0 & \\ 0 & -Y_{23} & & \ddots & \end{bmatrix} \tag{4.13a}$$

Let $$Z = \begin{bmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ 0 & 0 & & \ddots \end{bmatrix} \tag{4.13b}$$

Then

$$V' = V_1 - \frac{1}{2} Z I_1 \tag{4.14a}$$

$$I_2 = I_1 - AV' \tag{4.14b}$$

$$V_2 = V' - \frac{1}{2} Z I_2 \tag{4.14c}$$

Note that I_2 is defined in the same direction as I_1, which is contrary to the usual port conventions, but is necessary for defining consistent analogies between currents and the physical fields they are representing.

Combining the equations we have for the transmission matrix

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = {}_2T_1 \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \tag{4.15}$$

where

$${}_2T_1 \quad \begin{bmatrix} (E + \frac{1}{2} ZA) & -(Z + \frac{1}{4} ZAZ) \\ - A & (E + \frac{1}{2} AZ) \end{bmatrix} \tag{4.16}$$

The submatrices of T are all tri-diagonal. For networks involving cross-coupling these procedures are more complicated.

Greenfield Algorithm

When the measurement edge is the long dimension, or when sources are located throughout,many of the advantages of using transmission matrices are lost. It is still useful to group the network into rows or columns (whichever have the fewest nodes) and efficient solutions of (4.12d) can be obtained. When the network is divided up into rows or columns (4.12d) takes the form

$$YV = \begin{bmatrix} A_1 & C_{12} & 0 & & \\ C_{12} & A_2 & C_{23} & & O \\ 0 & C_{23} & \ddots & \ddots & \\ & & \ddots & \ddots & C_{mn} \\ & O & & C_{mn} & A_n \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ \vdots \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} J_1 \\ \vdots \\ \vdots \\ J_n \end{bmatrix} \tag{4.17}$$

The A matrices have the same form as (4.13a), except that the diagonal elements are augmented by the conductances connecting the node to its neighbor rows or columns. C_{ij} is a diagonal matrix whose elements are the negative of the conductances connecting the ith and jth rows or columns. V_i and J_i are the voltages and source currents at each node of the ith row or column.

Y can be decomposed into two triangular matrices

$$Y = EF \qquad E = \begin{bmatrix} I & 0 & 0 \\ E_{12} & I & 0 \\ 0 & E_{23} & I \end{bmatrix} \qquad F = \begin{bmatrix} F_1 & G_{12} & 0 & \cdots \\ 0 & F_2 & G_{23} & \\ \vdots & O & \ddots & \ddots \end{bmatrix} \tag{4.18}$$

$$\text{Let } FV = Z \tag{4.19}$$

Therefore (4.17) becomes $EZ = J$ (4.20)

To find E and F we have starting with $F_1 = A_1$

$$G_{ij} = C_{ij} \tag{4.20a}$$

$$E_{ij} = C_{ij}\, F_i^{-1} \tag{4.20b}$$

$$F_{i+1} = A_{i+1} - E_{ij}\, G_{ij} \tag{4.20c}$$

If E_{ij} and F_i^{-1} are saved we have an efficient procedure for determining the effects of any source configuration. Starting with $Z_1 = J_1$ we determine Z as

$$Z_j = J_j - E_{ij}\, J_i \tag{4.21}$$

V is now given by

$$V_n = F_n^{-1}\, Z_n \tag{4.22a}$$

$$V_i = F_i^{-1}(Z_i - G_{ij}\, V_j) \tag{4.22b}$$

If only one source configuration is involved (4.21) can be performed with (4.20) and only F_i^{-1} needs to be saved. When we consider the inverse boundary value problem the need for multiple source configurations will become apparent.

Chapter V. Telegen's Theorem and Applications

There exists an interesting and useful relationship in circuit analysis known as Telegen's Theorem. This is a very general relationship which leads to a wide range of applications. A recent book entitled "Telegen's Theorem and Electrical Networks" (Penfield, Spence and Duinker, 1970) explores a wide range of applications and demonstrates the versatility of the theorem. We will use their symbols and nomenclature. We are especially interested in the reciprocity relationships and the sensitivity analysis as these relate directly to the inverse boundary value problem. These relationship are also conceptually useful in understanding relationships of the physical systems which are modelled by networks.

Proof of Telegen's Theorem

Kirchoff's voltage law was given as

$$B^T V = 0 \tag{4.7}$$

From the branch-mesh relationships (4.5) and (4.6) we also have

$$I = BI_\varepsilon \tag{5.1}$$

Thus if we consider the sum of voltage drop-current products we can write from (5.1) and (4.7)

$$I^T V = I_\varepsilon{}^T B^T V = 0 \tag{5.2}$$

If we separate out those node pairs we will consider as ports and use the port convention for current and voltage which is opposite to that for circuit elements we have

$$\sum_p i_p V_p = \sum_\alpha i_\alpha V_\alpha \tag{5.3}$$

Current sources and voltage sources are taken care of by the ports.

This relationship is a topological one involving only the divergenceless nature of i and the curl free nature of V. It does not depend on the nature of the interconnecting elements, only on the arrangement of the connections. In fact it is not necessary for the currents and voltages to represent the actual currents and voltages of the network. i could be the current of one network and V the voltage drops of another network as long as the two networks had the same topology.

Thus we can generalize (5.3) by introducing Kirchoff current operators Λ' and Kirchoff voltage operators Λ'' which transform any currents and voltages which obey Kirchoff's laws into new currents and voltages which also obey these laws.

General Telegen Theorem $\sum_p \Lambda' i_p \Lambda'' v_p = \sum_\alpha \Lambda' i_\alpha \Lambda'' v_\alpha$ (5.4)

Reciprocity of Linear Networks

If we consider two different sets of currents (i^1, i^2) and voltages (v^1, v^2) in the same network we have from Telegen's theorem

$$\sum_p \left(i_p^1 v_p^2 - i_p^2 v_p^1 \right) = \sum_\alpha \left(i_\alpha^1 v_\alpha^2 - i_\alpha^2 v_\alpha^1 \right) \tag{5.5}$$

If the network elements are all linear and reciprocal

$$i_\alpha^1 v_\alpha^2 = i_\alpha^2 v_\alpha^1 \tag{5.6}$$

therefore
$$\sum_p i_p^1 v_p^2 = \sum_p i_p^2 v_p^1 \tag{5.7}$$

The relationship between the port current and voltages is the port admittance or impedance matrix

$$I_p = Y\, V_p \tag{5.8a}$$

$$V_p = Z\, I_p \tag{5.8b}$$

Let (i^1, v^1) represent the current and voltages when port j has a unit voltage and all the other ports are short circuited and let (i^2, v^2) represent the case with a unit voltage on i and all other ports short circuited. Thus we have from (5.8a)

$$i_i^{\,1} = Y_{ij}\ ,\quad i_j^{\,2} = Y_{ji}$$

and (5.7) gives
$$Y_{ij} = Y_{ji} \tag{5.9}$$

As an example let us consider the seismic network system as given in (3.22). The network currents are equal to the stress components times an area and $i_p v_p$ has the form

$$i_p v_p = u \cdot {}^n T\, \Delta S \tag{5.10a}$$

Circuit ground is used as the other node of each port pair. u is the vector displacement and ${}^nT$ is the stress vector distribution on the boundary element ΔS. Thus for instance on the x face $i_p v_p$ includes $V_x P_{xx}$, $V_y P_{yx}$, and $V_z P_{zx}$

If interior current sources are included and those nodes are also considered port nodes, the stress unbalance represented by the current source has the form of a body face.

$$i_s v_s = u \cdot F\, \Delta V \tag{5.10b}$$

where F is the body force and ΔV is the volume element. Thus a current source at a v_y node causes a net F_y on the volume represented by the node since that current causes unbalances in $P_{yx}\,\Delta y\Delta z$, $P_{yy}\,\Delta x\Delta z$, and $P_{yz}\,\Delta x\Delta y$.

If we write (5.7) in the limit of a finer and finer network representation we have

$$\iint [u_1 \cdot {}^{n}T_2 - u_2 \cdot {}^{n}T_1]\, dS = -\iiint [u_1 \cdot F_2 - u_2 \cdot F_1]\, dV \qquad (5.11)$$

This is the seismic reciprocity relationship for sinusoidal solutions (Gangi, 1970).

The two dimensional electromagnetic network system (3.28) can be shown to lead to an equivalent relationship

$$\iint (E_1 \times H_2 - E_2 \times H_1) \cdot dS = -\iiint \operatorname{div}(E_1 \times H_2 - E_2 \times H_1)\, dV \qquad (5.12)$$

This relationship is still valid in three dimensional problems even though our network analogy breaks down.

Green's functions and their reciprocity relationships can be obtained by considering point sources. In our seismic case, for instance, if we assume homogeneous boundary conditions and delta function body forces in (5.11) we have

$$G_{ij}(P, Q) = G_{ji}(Q, P) \qquad (5.13)$$

where $G_{ij}(P, Q)$ means the velocity component v_i at P due to a unit point force F_j at Q.

Sensitivity of Port Impedances

In the inverse boundary value problem one is attempting to determine interior properties on the basis of boundary measurements. One method of obtaining such solutions is to begin with a starting model and a differential analysis of how small changes in the model effect the boundary values. These derivatives are then used to estimate the model changes needed to reduce the errors between the model predictions and the measured boundary values. A perturbation of the model can be approximated by an appropriate source in an otherwise unperturbed model. In Chapter IV it was shown how for two dimensional networks one could efficiently solve for the effect of different sources, once a basic solution was obtained. This efficiency is further increased, however, if one makes use of reciprocity relations, for then it becomes necessary to solve for sources only at the measurement positions. These results can be arrived at directly from Telegen's theorem.

For small changes of impedance one can write

$$I^2 \delta Z \doteq I\, \delta(ZI) - \delta I\, (ZI) \qquad (5.14a)$$

$$\text{or} \quad I^2 \delta Z \doteq I\, \delta V - \delta I\, V \qquad (5.14b)$$

For multiport networks we have also

$$\sum_p\sum_q \Lambda^1 I_p \Lambda^2 I_q \,\delta Z_{pq} = \sum_p\sum_q \left[\Lambda^1 I_p \delta(\Lambda^2 I_q Z_{pq}) - \Lambda^2 \delta I_q \Lambda^1 I_p Z_{pq}\right] \tag{5.15a}$$

for symmetric Z_{pq} this becomes

$$\sum_p\sum_q \Lambda^1 I_p \Lambda^2 I_q \,\delta Z_{pq} = \sum_p \Lambda^1 I_p \Lambda^2 \delta V_p - \sum_q \Lambda^1 V_q \Lambda^2 \delta I_q \tag{5.15b}$$

Applying Telegen's theorem (5.4) to R.H.S.

$$\sum_p\sum_q \Lambda^1 I_p \Lambda^2 I_q \,\delta Z_{pq} = \sum_\alpha \Lambda^1 I_\alpha \Lambda^2 \delta V_\alpha - \sum_\beta \Lambda^1 V_\beta \Lambda^2 \delta I_\beta \tag{5.16}$$

Applying (5.15b) to R.H.S.

$$\sum_p\sum_q \Lambda^1 I_p \Lambda^2 I_q \,\delta Z_{pq} = \sum_\alpha\sum_\beta \Lambda^1 I_\alpha \Lambda^2 I_\beta \,\delta Z_{\alpha\beta} \tag{5.17}$$

If Λ^1 represents currents due to a unit port current at i and Λ^2 represents a unit port current at j and if $Z_{\alpha\beta}$ is diagonal (no transformers) then

$$\delta Z_{ij} = \Lambda^1 I_\alpha \Lambda^2 I_\alpha \,\delta Z_\alpha \tag{5.18a}$$

or

$$\delta \ln Z_{ij} = (Z_\alpha / Z_{ij}) \Lambda^1 I_\alpha \Lambda^2 I_\alpha \,\delta \ln Z_\alpha \tag{5.18b}$$

Equation (5.18) shows us that if we are only interested in port impedances at m ports, the sensitivity matrix can be determined by looking at the network solutions for m excitations.

Equations (5.16) and (5.17) could be reorganized to give

$$\sum_p\sum_q \Lambda^1 I_p \Lambda^2 I_q \,\delta Z_{pq} = -\sum_\alpha\sum_\beta \Lambda^1 V_\beta \Lambda^2 V_\alpha \,\delta Y_{\alpha\beta} \tag{5.19}$$

thus

$$\delta Z_{ij} = -\Lambda^1 V_\alpha \Lambda^2 V_\alpha \,\delta Y_\alpha \tag{5.20}$$

$\Lambda^1 V_\alpha$ can be represented as $Z'_{\alpha i}$ and $\Lambda^2 V_\alpha$ as $Z'_{\alpha j}$

thus

$$\delta Z_{ij} = -Z'_{\alpha i} Z'_{\alpha j} \,\delta Y_\alpha \tag{5.21}$$

$Z'_{\alpha i}$ can be considered a port impedance simply by placing a port across the α branch. If ports are defined across every branch we can also define $Z'_{\alpha\beta}$ as the voltage induced across the α branch by current sources placed across the β branch. Thus from (5.21) we have

$$\delta Z'_{\beta i} = -Z'_{\alpha\beta} Z'_{\alpha i} \,\delta Y_\alpha \tag{5.22}$$

From (5.21) and (5.22) we can write

$$\partial^2 Z_{ij} / \partial Y_\alpha \partial Y_\beta = Z'_{\beta\alpha} Z'_{\beta i} Z'_{\alpha j} + Z'_{\alpha i} Z'_{\beta\alpha} Z'_{\beta j} \tag{5.23}$$

This can be extended to higher derivatives. To use this formulation, however, the network must be solved for current sources across every branch, not just the ports at which we wish to determine the impedance changes.

Chapter VI. Inverse Boundary Value Problem.

In the past most of the efforts in dealing with inverse boundary value problems have been concerned with methods for finding a solution. For some problems, such as the seismic refraction problem in a region of increasing velocity with depth, formal solutions have existed for a long time. More recently it has become known that a whole class of one dimensional reflection or scattering inverse problems are formally solvable by the Gel'fand-Levitan algorithm (I.M. Gel'fand and B.M. Levitan, 1955). Most inverse problems are solved by iteration schemes, however. The iteration method exposes an important aspect of the inverse problem, the non uniqueness of the solution or the resolving power of the data. Our understanding and appreciation of these factors have been greatly aided by the pioneering work of Backus and Gilbert (Backus, G.E., and J.F. Gilbert, 1967, 1968, and 1970). A recent excellent review of this approach using matrix notation is given by Wiggins (R.A. Wiggins, 1972).

Direct methods for solving inverse boundary value problems in two or more dimensions are still lacking, but the iteration schemes are in principal not limited. Equation (5.18) shows us that one can determine a sensitivity matrix efficiently even for two dimensional problems which makes the iteration scheme quite practical.

If our observed field gives us Z_{ij}^{obs} and our model predicts Z_{ij}^{model}, then we can make a correction to the model by solving the equations

$$\left[\partial Z_{ij} / \partial Z_\alpha\right] \Delta Z_\alpha = Z_{ij}^{obs} - Z_{ij}^{model} \tag{6.1}$$

For most problems it is best to deal with logarithmic variations as this weights the data and parameters uniformly

$$\left[\partial \ln Z_{ij} / \partial \ln Z_\alpha\right] \Delta \ln Z_\alpha = \ln\left(Z_{ij}^{obs} / Z_{ij}^{model}\right) \tag{6.2}$$

If our model is very close to correct (6.2) should lead to a good fit. In general one has to iterate this procedure by recomputing a new set of Z_{ij}^{model} values and a new sensitivity matrix, [$\partial \ln Z_{ij} / \partial \ln Z_\alpha$], using the new model values Z_α. Thus this procedure for solving the inverse problem is not a linear procedure. Nevertheless some very important insights into the nature of the inverse problem can be gained by studying the properties of the linear system (6.1) or (6.2).

General Matrix Analysis

Following closely Lanczos (C. Lanczos, 1961), consider an n x m matrix A (n rows, m columns) and the solution of the matrix equation

$$Ay = b \tag{6.3}$$

From A form an expanded Hermitian matrix

$$S = \begin{bmatrix} 0 & A \\ \widetilde{A} & 0 \end{bmatrix} \qquad \text{where } A = (A^{*})^{T}$$

The eigenvalues of S must be real, and the eigenvectors satisfy the coupled equations

$$A v_i = \lambda_i u_i \tag{6.4a}$$

$$\widetilde{A} u_i = \lambda_i v_i \tag{6.4b}$$

If $\begin{bmatrix} v_i \\ u_i \end{bmatrix}$ satisfy (6.4) for eigenvalue λ_i

then $\begin{bmatrix} v_i \\ -u_i \end{bmatrix}$ satisfy (6.4) for eigenvalue $-\lambda_i$

Also since

$$\begin{bmatrix} \widetilde{v}_i & \widetilde{u}_i \end{bmatrix} \begin{bmatrix} v_j \\ u_j \end{bmatrix} = \delta_{ij} \tag{6.5}$$

and

$$\begin{bmatrix} \widetilde{v}_i & \widetilde{u}_i \end{bmatrix} \begin{bmatrix} v_j \\ -u_j \end{bmatrix} = 0 \tag{6.5b}$$

we have from (6.5a) + (6.5b)

$$\widetilde{v}_i v_j = \delta_{ij} \qquad \text{(renormalized)} \tag{6.6a}$$

and from (6.5a) - (6.5b)

$$\widetilde{u}_i u_j = \delta_{ij} \tag{6.6b}$$

The (m + n) eigenvectors of S can be arranged in the following order

$$\begin{array}{llll} \lambda_i, & v_i, & u_i & \qquad i = 1 \cdots p \\ -\lambda_i, & v_i, & -u_i & \qquad i = 1 \cdots p \\ 0, & v_i^{(0)}, & 0 & \qquad i = 1 \cdots m-p \\ 0, & 0, & u_i^{(0)} & \qquad i = 1 \cdots n-p \end{array} \tag{6.7}$$

p is the rank of matrix A(and $\widetilde{A}$)

From (6.4) we see that λ_i^2 are eigenvalues for $A\widetilde{A}$ and $\widetilde{A}A$ which are n x n and m x m Hermitian matrices.

Therefore $p \leq m$, $\leq n$

We separate out the eigenvectors associated with non zero eigenvalues from those associated with zero eigenvalues and consider two spaces for V and U

$$
\begin{aligned}
V_p \text{ space} &= [v_1 \cdots v_p] \\
V^{(0)} \text{ space} &= [v_1^{(0)} \cdots v_{m-p}^{(0)}]
\end{aligned}
\tag{6.8a}
$$

$$
\begin{aligned}
U_p \text{ space} &= [u_1 \cdots u_p] \\
U^{(0)} \text{space} &= [u_1^{(0)} \cdots u_{n-p}^{(0)}]
\end{aligned}
\tag{6.8b}
$$

The matrix A destroys $V^{(0)}$ space vectors, therefore the solution Ay = b is <u>non unique</u> if any $V^{(0)}$ exists.

The matrix A cannot create any $U^{(0)}$ space vectors, therefore (6.3) is <u>incompatible</u> if b is comprised of any $U^{(0)}$ space. This poses real problems in seeking solutions to (6.3). One way out is the generalized matrix inverse. One can take care of non uniqueness by many methods, but one natural way is to consider the solution that is devoid of any $V^{(0)}$ space vectors. For our inverse boundary value problem this is equivalent to finding a solution to (6.2) that involves the least change in the model.

No exact solution can be obtained if b has any $u^{(0)}$ vectors, but a smallest error solution in the least squared sense is obtained by solving

$$Ay = b' \quad \text{where} \quad b' = b - [U^{(0)}][\tilde{U}^{(0)}]\,b \tag{6.9}$$

i.e. b' is projection of b on U_p space

$$\text{Thus if} \quad b = \sum_1^p b_i' u_i + \sum_1^{n-p} b_i^{(0)} u_i^{(0)} \tag{6.10}$$

the generalized solution is

$$y' = \sum_1^p b_i' v_i / \lambda_i$$

or

$$y' = V_p \Lambda_p^{-1} \tilde{U}_p\, b \tag{6.11}$$

where

$$\Lambda_p = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}$$

Since (6.4a) can be written

$$A V_p = U_p \Lambda_p \tag{6.12}$$

and (6.6b) can be written

$$\tilde{U}_p U_p = E_p \text{ (identity matrix)} \tag{6.13}$$

we have

$$\Lambda_p = \tilde{U}_p A V_p \tag{6.14}$$

and our generalized inverse operator can be written as

$$\text{A inverse} = V_p (\tilde{U}_p A V_p)^{-1} \tilde{U}_p \tag{6.15}$$

The squelching of the $V^{(0)}$ space is especially important in the inversion scheme for the improvement in the fit implied by solving (6.2) will only hold up if the changes made in the model are small. If $V^{(0)}$ is not excluded strongly, arbitrarily large model changes might result and (6.2) would not be accurate. As an example suppose $U_i^{(0)}$ represented a change in the thickness of a thin surface layer which was too thin to influence the data. If we did not suppress $V^{(0)}$ we might make this layer so thick that it influenced the model results and the new model could be worse than the original model.

Even non zero eigenvalues can be troublesome if they are too small. Let us choose some arbitrary dividing line between large and small eigenvalues, $\lambda_q \geq \epsilon > \lambda_\epsilon$

$$\Lambda_p = \begin{bmatrix} \Lambda_q & 0 \\ 0 & \Lambda_\epsilon \end{bmatrix}, \quad V_p = \begin{bmatrix} V_q & V_\epsilon \end{bmatrix}, \quad U_p = \begin{bmatrix} U_q & U_\epsilon \end{bmatrix} \tag{6.16}$$

$$y' = y_q + y_\epsilon = \left[V_q \Lambda_q^{-1} \tilde{U}_q + V_\epsilon \Lambda_\epsilon^{-1} \tilde{U}_\epsilon \right] b \tag{6.17}$$

If b has any appreciable b_ϵ, y_ϵ would be too large for (6.2) to work. Thus a more practical inverse operator would be

$$\text{A inverse} = V_q (\tilde{U}_q A V_q)^{-1} \tilde{U}_q \tag{6.18}$$

The optimum size of ϵ depends on the nature of the problem and the nature of the data. The use of (6.2) rather than (6.1) is helpful, however, in bringing some uniformity to the procedure for determining ϵ. Because of the homogeneous nature of the network equations the row sums of the matrix $[\partial \ln z_{ij} / \partial \ln z_\alpha]$ add up to unity. In most cases the terms are also all positive. It can then be shown that the max $\lambda \leq 1$. In problems where large errors in the data or in the model justification exist, a modest cut-off at $\epsilon = 0.25$ may be necessary. When close fits to the data are being achieved, this restriction can be relaxed somewhat.

(6.18) requires one to make a complete eigenvector analysis. An approximation to (6.18) can be arrived at very simply however. Since $\tilde{A}$ destroys $U^{(0)}$, the least square error problem can be stated as

$$\tilde{A} A y = \tilde{A} b \tag{6.19}$$

The $V^{(0)}$ space problem can be circumvented by considering $\tilde{A}A + \epsilon^2 E_m$. This has the same eigenvectors $V_p, V^{(0)}$ as $\tilde{A}A$ but the eigenvalues become $\lambda_i^2 + \epsilon^2$

Thus if
$$(AA + \epsilon^2 E)\, y'' = \hat{A} b$$

then
$$y'' = \sum_1^p \frac{\lambda_i b_i}{\lambda_i^2 + \epsilon^2}\, v_i \tag{6.20}$$

Comparing (6.20) to (6.11) we find

$$y_i'' = \left[\lambda_i^2/(\lambda_i^2 + \epsilon^2)\right] y_i' \tag{6.21}$$

The solution (6.20) is obtained from the operator

$$\text{A inverse} = (\tilde{A}A + \epsilon^2 E)^{-1} \tilde{A} \tag{6.22}$$

This operator eliminates the effects of $U^{(0)}$ and $V^{(0)}$ and reduces the effects of V_ϵ and U_ϵ and is thus similar to but not exactly equal to (6.18).

Finding a solution is only part of the inverse problem, one must also have some idea about the range of possible solutions. This information is also contained in the inverse operator in as far as the linearization is valid.

Let Y'' be our particular solution and y be any possible solution in the least squares sense.

Thus $Y'' = (\tilde{A}A + \epsilon^2 E)^{-1} \tilde{A} b$ (6.23)

and $\tilde{A} A y = \tilde{A} b$ (6.24)

Combining (6.23) and (6.24) we have

$$y'' = Cy \tag{6.25}$$

$$C = [\text{A inverse}]A \tag{6.26}$$

Thus the particular solution is obtainable by a single matrix operator from any of the possible solutions. The nature of this operator C gives us the desired information about the resolution of our model. If a row of C is essentially like a row of E, i.e. if the diagonal element is close to one and the other elements are close to zero, then that parameter is well resolved and unique. If the row has power, but it is distributed among several elements, that parameter is poorly resolvable from the data and only average properties are determined. If the row is essentially null, that parameter is not represented in the data.

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