DISCUSSION

Discussion on "The magnetotelluric method in the exploration of sedimentary basins" by Keeva Vozoff (GEOPHYSICS, February 1972, p. 98–114).

A number of errors in the section dealing with the Tipper have been pointed out. Some alternative definitions and additional properties have also become apparent.

Equations (37), (39), (40), and (41) are incorrect, and are corrected or replaced below. A sign error occurs in equation (26); it should read

$$\tan 4\theta = \frac{(Z_{xx} - Z_{yy})(Z_{xy} + Z_{yx})^* + (Z_{xx} + Z_{yy})^*(Z_{xy} - Z_{yx})}{|Z_{xx} - Z_{yy}|^2 - |Z_{xy} + Z_{yx}|^2}.$$

(1) *Rotation Conventions*

For a two-dimensional (2-D) earth with x' direction across strike, we know that

$$H_z = A' H_{x'},$$

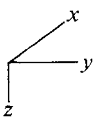
where A' is a complex transfer function—the "Tipper".

For an arbitrary coordinate system, we have

$$H_z = A\ H_x + B\ H_y.$$

Let

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$



Q is a rotation of the (x,y) axes clockwise from the x direction (or "north") to the y direction (or "east") with z down.

$$H_z = [A\ \ B]\begin{bmatrix} H_x \\ H_y \end{bmatrix} = \left[Q\begin{bmatrix} A \\ B \end{bmatrix}\right]^T Q\begin{bmatrix} H_x \\ H_y \end{bmatrix}$$

$$= [A'B']\begin{bmatrix} H_{x'} \\ H_{y'} \end{bmatrix},$$

where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = Q\begin{bmatrix} x \\ y \end{bmatrix}$$

are the rotated coordinates.

(For a 2-D earth we know $B' = -A\sin\theta + B\cos\theta = 0$ for some θ, and the Tipper analysis is a study of 2-D effects in the data.)

(2) *Estimation of A, B, Tipper coherency, and residual error*

This is done by least-squares in a similar way to the z_{ij}, i.e.,

$$\begin{cases} \langle H_z, \bar{H}_x\rangle = A\langle H_x, \bar{H}_x\rangle + B\langle H_y, \bar{H}_x\rangle \\ \langle H_z, \bar{H}_y\rangle = A\langle H_x, \bar{H}_y\rangle + B\langle H_y, \bar{H}_y\rangle \end{cases}$$

where $<>$ denotes a band average, and $\bar{H}_x$ is the complex conjugate of H_x.

This gives

$$A = \frac{1}{\text{Botl}}\frac{\langle H_z, \bar{H}_z\rangle^{1/2}}{\langle H_x, \bar{H}_x\rangle^{1/2}}[\text{Coh}\ (H_z, H_x) - \text{Coh}\ (H_z, H_y)\ \text{Coh}\ (H_y, H_x)],$$

and

$$B = \frac{1}{\text{Botl}}\frac{\langle H_z, \bar{H}_z\rangle^{1/2}}{\langle H_y, \bar{H}_y\rangle^{1/2}}[\text{Coh}\ (H_z, H_y) - \text{Coh}\ (H_z, H_x)\ \text{Coh}\ (H_x, H_y)]$$

where

$$\text{Botl} = (1 - |\text{Coh}\ (H_x, H_z)|^2),$$

and

$$\text{Coh}\ (H_x, H_y) = \frac{\langle H_x, \bar{H}_y\rangle}{|H_x|\ |H_y|}.$$

$H_z^p = A H_x + B H_y$ is the predicted H_z due to the horizontal field. How well the H_z field can be estimated from the (H_x, H_y) fields is measured by:

(A) *Tipper coherency*

$$\mathrm{Coh}\,(H_z, H_z^p) = \frac{p\,\mathrm{Coh}\,(H_z, H_x) + q\,\mathrm{Coh}\,(H_z, H_y)}{\{|p|^2 + |q|^2 + 2\,\mathrm{Re}\,[\bar{p}q\,\mathrm{Coh}\,(H_x, H_y)]\}^{1/2}}$$

where

$$p = \bar{A}\langle H_x, \bar{H}_x\rangle^{1/2}$$

$$q = \bar{B}\langle H_y, \bar{H}_y\rangle^{1/2}$$

The important property of this coherency is that it is *Real,* i.e., the phase difference between H_z and H_z^p is zero.

$$\delta = \frac{(a_r^2 + a_i^2)\arctan(a_i/a_r) + (b_r^2 + b_i^2)\arctan(b_i/b_r)}{T^2}\ (-90° \leq \delta \leq 90°).$$

(B) *Residual error*

$$\mathrm{Res} = \frac{\langle H_z - H_z^p, \bar{H}_z - \bar{H}_z^p\rangle^{1/2}}{\langle H_z, \bar{H}_z\rangle^{1/2}}$$

$$= (1 - R^2/\langle H_z, \bar{H}_z\rangle)^{1/2}\ {*}100(\%)$$

where $R = \{|p|^2 + |q|^2 + \mathrm{Re}\,[\bar{p}\,q\,\mathrm{Coh}\,(H_x, H_y)]\}^{1\ 2}$ [i.e., the bottom line of expression (1)]. This result uses the least-squares property that

$$\mathrm{Res}^2 = (1 - \langle H_z^p, \bar{H}_z^p\rangle/\langle H_z, \bar{H}_z\rangle)$$

The residual error is the percentage residual H_z field not predicted by (A, B). If H_z is due in part to the source, as is observed at longer periods, then Tipper coherency may be reduced and Res enhanced.

(3) *Tipper, tipper phase, and tipper skew*

If $\mathrm{Coh}\,(H_z, H_z^p)$ is ÷ 1, the situation need not be 2-D. For a two-dimensional Tipper, we need some rotation angle ϕ where $|B|^2$ vanishes, or *nearly* vanishes.

Three measures that are independent of rotation and provide information about structure are:

$$\phi_1 = \frac{(a_r^2 + b_r^2)\arctan(b_r/a_r) + (a_i^2 + b_i^2)\arctan(b_i/a_i)}{T^2},$$

(A) *Tipper*

$$T = (a_r^2 + a_i^2 + b_r^2 + b_i^2)^{1\ 2},$$

where a_r, a_i and b_r, b_i are real and imaginary parts of A and B, respectively. Tipper is independent of rotation ϕ. If there is some ϕ for which

$$H_{xp} = A'H_{x1},$$

then

$$T^2 = |A'|^2.$$

(B) *Tipper phase*

δ is rotation invariant, and if the earth is 2-D, then δ is the phase of A'.

(C) *Tipper "skew"*

A skew is a rotation invariant measure which is zero for a 2-D earth. We can use

$$\mathrm{Skew} = \frac{2(a_r b_i - a_i b_r)}{T^2}.$$

This is zero for a 2-D earth.

(4) *Estimates of the rotation and angle ϕ*

For the 2-D earth

$$A = A'\cos\phi \text{ and } B = A'\sin\phi.$$

That is,

$$\phi = \arctan(b_r/a_r)$$

$$= \arctan(b_i/a_i)$$

$$= \arctan(B/A) \text{ for exact 2-D data)}$$

However, these are subject to wide variations. We can choose a weighted estimate,

as in the Vozoff paper.

A second estimate of ϕ is the natural extension of the tensor rotation method to Tipper, and was proposed by Sims and Bostick (1969). Here ϕ is chosen to maximize

$$|A'|^2 = |A \cos\phi + B \sin\phi|^2,$$

and is denoted ϕ_2.

We find the necessary condition to be,

$$\tan(2\phi_2) = \frac{2(a_r b_r + a_i b_i)}{(a_r^2 + a_i^2) - (b_r^2 + b_i^2)}.$$

If we use the Fortran function *ATAN* so that -90 degrees $\leq 2\phi_2 \leq 90$ degrees, then $|A'|^2$ is a *maximum* if $(a_r^2 + a_i^2) \geq (b_r^2 + b_i^2)$, and $|A'|^2$ is a *minimum if* $(a_r^2 + a_i^2) \leq (b_r^2 + b_i^2)$. If $|A'|^2$ is minimum, we need to add 180 degrees to $2\phi_2$. In either case, the final angle ϕ_2 is determined to within 180 degrees.

A definite choice for ϕ_2 is made by taking ϕ_2 as the angle for which the phase of A' is in the range (-90 degrees, 90 degrees). (The reason for this choice is described in the next section.)

For a 2-D earth,

$$\phi_1 \div \phi_2,$$
$$\text{Skew} \div 0,$$

and

$$|B'|^2 \div 0.$$

(5) *Coh* $(H_z, H_{x'})$ *and Tipper "direction"*

Coh $(H_z, H_{x'})$ is no harder to get than Coh (H_z, H_z^P). That is,

$$\begin{aligned}\langle H_z, \cos\phi \bar{H}_x + \sin\phi \bar{H}_y\rangle &= \cos\phi\langle H_z, \bar{H}_x\rangle + \sin\phi\langle H_z, \bar{H}_y\rangle \\ &= \cos\phi\, |H_z|\, |H_x| \text{ Coh } (H_z, H_x) \\ &\quad + \sin\phi\, |H_z|\, |H_y| \text{ Coh } (H_z, H_y)).\end{aligned}$$

Let $r = \cos\phi|H_x|$, $s = \sin\phi|H_y|$.

Then,

$$\text{Coh } (H_z, H_{x'}) = \frac{r \text{ Coh } (H_z, H_x) + s \text{ Coh } (H_z, H_y)}{\{r^2 + s^2 + 2rs \text{ Re } [\text{Coh } (H_x, H_y)]\}^{1/2}}$$

The phase of Coh $(H_z, H_{x'})$ is the phase lag between H_z and the (rotated) $H_{x'}$, and since Coh (H_z, H_z^P) has zero phase [cf., section (2)], we have, for a 2-D earth,

$$\begin{aligned}\text{Coh } (H_z, H_z^P) &= \text{Coh } (H_z, A'H_{x'}) \\ &= \frac{\bar{A}'}{|A'|} \text{ Coh } (H_z, H_{x'}).\end{aligned}$$

That is,

$$|\text{Coh } (H_z\ H_x')| = |\text{Coh } (H_z H_z^P)|,$$

and the phase of Coh $(H_z, H_{x'})$ is the phase of A', (which is $\delta \pm 180$ degrees), for a 2-D earth.

The phase of Coh $(H_z, H_{x'})$ (and therefore of A') may be chosen in the range (-90 degrees, 90 degrees) since the rotation angle ϕ obtained from section (4) is unique up to 180 degrees, and ϕ so determined is the Tipper "direction". (This choice of Tipper direction is motivated by the fact that, for a vertical contact, the tipping vector will be pointing downward from the conductive to the resistive side of the contact.)

(6) *The effects of errors*

Suppose the earth is 2-D, but that there is noise on the H_x and H_y channels.

The $\left\{\begin{matrix} H_x \\ H_y \end{matrix}\right.$ noise is assumed to be incoherent with $\left\{\begin{matrix} H_y \\ H_x \end{matrix}\right.$ and H_z, and the noise powers are denoted $|n_x|^2$ and $|n_y|^2$.

In this case, we know that

$$A = t_x A,$$

and

$$B = t_y B,$$

where

$$t_x = \left(\frac{|H_x|^2}{|H_x|^2 + |n_x|^2}\right)^{1/2},$$

and

$$t_y = \left(\frac{|H_y|^2}{|H_y|^2 + |n_y|^2}\right)^{1/2}.$$

The Tipper, and the Tipper coherency are reduced, and the residual error increased by the noise. However,

$$\text{Skew} \to t_x t_y * \text{Skew},$$

which means that if the skew is near zero, it cannot be increased by the type of noise we have introduced. The most important effect is the bias introduced into the rotation angles.

Let

$$u = t_x/t_y.$$

Then

$$\phi_1 = \arctan(1/u \tan\phi) \text{ for a } \textit{2-D earth},$$

and

$$\phi_2 = \tfrac{1}{2}\arctan(1/t \tan 2\phi),$$

where

$$t = \frac{(u^2+1)\cos 2\phi + (u^2-1)}{2u\cos 2\phi}.$$

If ϕ is close to zero, or 90 degrees, the bias is about the same. However, for general angles nearer to 45 degrees, the two estimates tend to diverge in opposite directions from ϕ.

For example, if

$$\phi = 45 \text{ degrees},$$

$$\phi_1 = \arctan(1/u),$$

and

$$\phi_2 = \tfrac{1}{2}\arctan\frac{2u}{(1-u^2)}.$$

If we tabulate the results we get

u	ϕ_1	ϕ_2	$\bar{\phi}\left(=\frac{\phi_1+\phi_2}{2}\right)$
2	26.6	63.4	45
1.5	33.7	56.3	45
1	45	45	45
.667	56.3	33.7	45
.5	63.4	26.6	45

That is,

$$\begin{cases} \phi_1(u) = \phi_2(1/u) \\ \text{and} \\ \dfrac{(\phi_1+\phi_2)}{2} = \phi \end{cases}$$

These results suggest that plotting both estimates would give an indication of divergence due to noise, provided the *skew* is low.

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REFERENCES

Sims, W. E., and Bostick, F. X. Jr., 1969, Methods of magnetotelluric analysis: EGRL Tech. rep. no. 58., University of Texas at Austin.