DIRECT AND INVERSE PROBLEMS IN LOCAL ELECTROMAGNETIC INDUCTION

Z. TARLOWSKI

Institute of Geophysics Polish Academy of Science, 00–973 Warsaw, Pasteura 3, Poland

Abstract. Various methods of solving direct and inverse problems in local electromagnetic induction are presented. In the section dealing with direct problems some improvments are suggested in the finite difference method in the case of two-dimensional modeling. Two ways of dealing with inverse problems are presented, the first continous, the other parametric. Emphasis is laid upon algebraic aspects of dealing with one-dimensional inverse problems.

1. Introduction

The main task of mathematical modelling in electromagnetic induction is to discover the distribution of the Earth's electrical parameters.

In direct problems we assume the shape of the primary sources and the distribution of electrical parameters of the model to be known. What we are looking for is the value of the vectors of the electromagnetic field in the whole space.

In inverse problems, the distribution of the model's electrical parameters is obtained from observation of the electromagnetic field of the Earth. In all the inverse methods used to date, the main problem is the ability to solve the direct problems effectively, either analytically or numerically.

We are approaching a scientific description of the algorithm which, being based on input data, would automatically give handy and reliable information about the distribution of the Earth's electrical parameters. The ideal situation would be when, knowing the precise shape of the primary sources, we could fix the distribution of the Earth's electrical parameters three-dimensionally. Despite the significant progress made towards solving these problems, the present situation is far from ideal.

Only direct one-dimensional problems can be solved analytically. Real direct two- and three-dimensional problems must be solved numerically. Some progress has been made in solving direct two-dimensional problems. One can observe two tendencies. The first depends on the improvement of the boundary conditions of the problem. The other is based on the improvment of the methods for solving the systems of linear equations resulting from the discretization of the problems.

Progress has also been made in three-dimensional modeling (Dey, 1978; Hohman, 1978, Pridmore, 1978; Dawson and Weaver, 1979), but we are not yet in a position to apply this practically in direct and inverse problems.

The characteristics of a given method when applied in practice should be effectivness and flexibility. It seems that even the solutions of two-dimensional problems are not always satisfactory, and it is perhaps useful to draw attention to the practicality of some of the methods.

Geophysical Surveys 4 (1982) 395–404. 0046–5763/82/0044–0395$01.50.

2. Direct problems

Let us look first at a direct two-dimensional problem, mentioning particularly the numerical methods. For a numerical solution to this problem there are several available methods e.g. the finite difference method, the finite element method or the integral equations method.

Of these, the most practical in two-dimensional direct problems seems to be the finite difference method, which enables a set of linear equations to be formed easily for solving the problem. When using the method of finite elements for an irregular grid, problems arise in producing flexible and effective input data for the computer program. On the other hand, in the integral equation method Green's function can not always be easily found for a wide range of problems.

Many researchers have been trying to improve the effectiveness and precision of the finite difference method.

In order to explain the electromagnetic induction for time-harmonic fields in two dimensions we must solve the boundary value problem for Helmholtz's equation.

For an E-polarization case we have:

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = i\xi E_x \cdot \tag{1}$$

For a B-polarization case we have:

$$\frac{\partial}{\partial y}\left(\eta \frac{\partial B_x}{\partial_y}\right) + \frac{\partial}{\partial_z}\left(\eta \frac{\partial B_x}{\partial_z}\right) = i\mathrm{B}_x \tag{2}$$

ξ and η depend on the physical parameters of the medium and the frequency of the source of the field.

After integrating Equations (1) and (2) at each mesh point (Varga, 1962) and employing an irregular five-point difference scheme to their new forms, Tarlowski (1978) obtained a set of linear equations with a symmetrical, non-hermitian matrix A, with complex elements.

$$A = \mathrm{Re}\, A + i\, \mathrm{Im}\, A \tag{3}$$

Matrix A is such that its real part Re A is a positive-definite matrix which is fixed, enabling us to express matrix A as:

$$\mathrm{Re}\, L(\mathrm{Re}\, L)^T = \mathrm{Re}\, A + \mathrm{Im}\, L(\mathrm{Im}\, L)^T, \tag{4}$$

Where L has the following form:

$$\begin{bmatrix} L_1 & & & \\ U_1 & L_2 & & \\ & U_2 & L_3 & \\ & \cdot & \cdot & \\ & \cdot & \cdot & \\ & & U_{N-1} & N_N \end{bmatrix}$$

L_l ($l = 1, 2, ..., N$) a lower triangular matrix (MxM)
U_l ($l = 1, 2, ..., N$-1) an upper triangular matrix (MxM)
N and M size of gird (Wieladek *et al.*, 1980).

Matrix A is a sparse and large matrix. These systems of equations are often solved with the use of iterative methods, which are, however, not always effective since matrix A is non-hermitian. In many cases straight methods turn out to be more effective for solving such systems. From the decomposition of matrix A by the Cholesky-Banachiewicz method we can write down the finite algorithm of the solution to the set of linear equations which requires $2M^3N+O(M^2N)$ multiplications. While the Gauss-Seidel iteration method requires $M^2/6$ iterations for the same number of multiplications, which is usually insufficient for the required reduction of the norm of the residual vector.

The numerical tests show the great effectiveness of this algorithm. For a problem

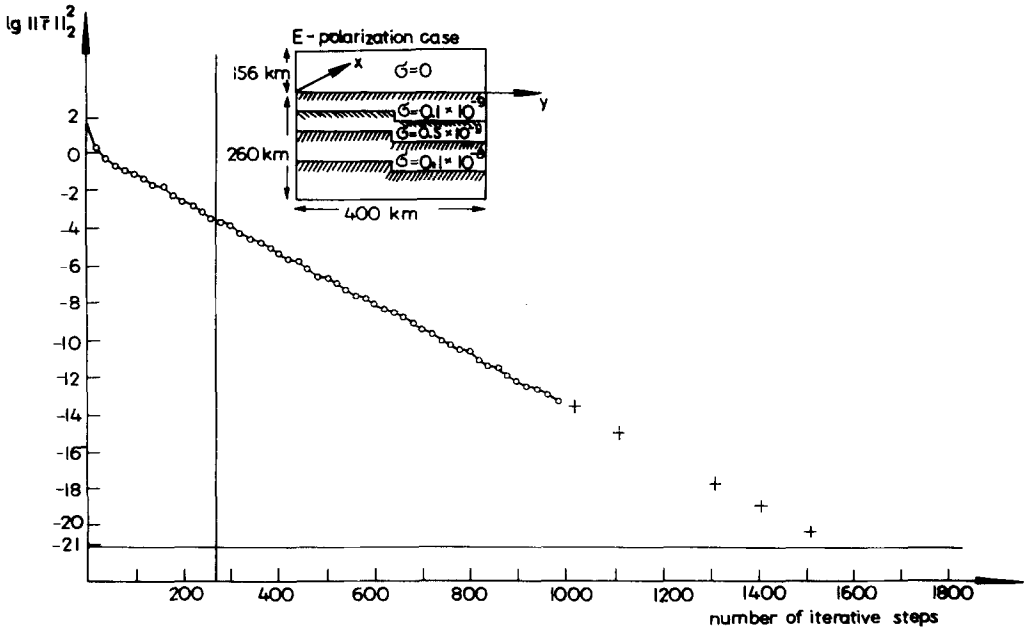

Fig. 1. The numerical test of the Gaus-Seidel and Cholesky-Banachiewicz methods (Wieladek *et al.*, 1980).

where M=39, N=39 and a model as in Figure 1, the reduction of the residual vector norm in the Cholesky-Banachiewicz method was 0.36×10^{-21}, while in the same number of operations it was only 0.5×10^{-4} in the Gauss-Seidel method.

Brewitt-Taylor and Johns (1977) used the diakoptic solution of two-dimensional induction problems. In this case the problem was divided into segments which were partially solved separately and then joined together by the application of boundary conditions at the edges of the segments. The advantages were: (a) a large problem was reduced to more manageable pieces, (b) in the solution of similiar problems only those segments whose conductivity was different needed be re-solved.

To find an asymptotic expression of an electric field in a non-conducting region and over long distances starting at the beginning of a co-ordinate system, Weaver and Brewitt-Taylor (1978) used improved conditions in an E-polarization case.

In an E-polarization case, the precision of the solution was increased by using a smaller approximating grid (Figure 2).

None of the methods at present available for solving electromagnetic induction problems in the Earth can be considered satisfactory when we assume conductivity to be a function of three variables. The finite difference method usually requires a lot of computer time and mass storage, and allows the analysis of simple three-dimensional structures (Jones and Vozoff, 1978).

A more effective solution is obtained when formulating the problem in vector integral equation terms (Weideldt, 1975), but it is somewhat restrictive in its applicability since it requires the anomalous region to be surrounded by a 'normal' i.e. layered structure.

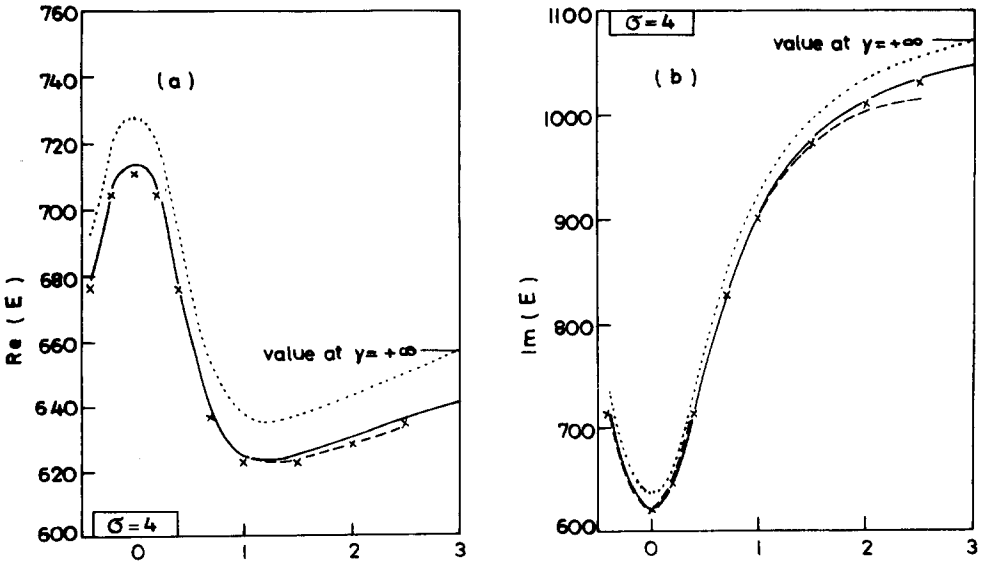

Fig. 2. The real and imaginary parts of the electric field at the surface of the conductor. The solid (———) and dotted (.....) lines show the fields calculated on the large and small grids for standard boundary conditions. The broken line (– – – –) shows the field calculated on the small grid with the improved boundary conditions (Weaver and Brewitt-Taylor, 1978).

In many cases, particularly where the channelling of induced electric currents in the oceans around islands and other geographic features near a coastline are taken into consideration, the region of laterally varying conductivity is effectively confined to a thin layer at the surface of the Earth. The conditions under which this approximation is valid have been examined by Weaver (1979).

A new method for solving problems in three-dimensional induction, in which the Earth is represented by a uniformly conducting half-space overlain by a surface layer of variable conductance was presented by Dawson and Weaver (1979).

Unlike previous treatments of this type of problem, the method does not require the fields to be separated into their normal and anomalous parts, nor is it necessary to assume that the anomalous region is surrounded by a uniform structure; the model may approach either an E- or B-polarization configuration at infinity.

The solution is expressed as a vector integral equation in the horizontal field at the surface of the Earth.

3. Inverse Problems

The inverse problems are often ill-posed, which means that small changes in the input data lead to big changes in the solution. Inverse problems are nonlinear. Various methods are now used to solve such problems. One can mention here gradient, Marquardt, Monte Carlo or linearization methods.

In recent years the linearization method has attracted much attention, and therefore we want to discuss here its mathematical aspects.

We would like to present here two different approaches in formulating the problem of finding the distribution of electric conductivity of the Earth in the one-dimensional case, when the observations of the electromagnetic field at the Earth's surface are given.

The first approach, let us call it parametric inversion, can be found in many works, e.g. Jupp and Vozoff (1975a).

The Earth models are determined by N free parameters, which we can write as the vector $\mathbf{x} = (x_1, x_2, ..., x_N)^T$.

The input data are based on M observations, which can be written as the vector $\mathbf{d}=(d_1, d_2, ..., d_M)^T$.

The direct problem generates a set of model data for each $\mathbf{x}$, which can be expressed as a vector function as follows:

$$\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), ..., g_M(\mathbf{x}))^T.$$

where $g_i(\mathbf{x})$ is the value predicted by the model, and corresponds to the observation d_i.

The inverse problem is to find values of $\mathbf{x}$ such that it matches $\mathbf{d}$ in some sense, which in this case is the minimum of the root mean square relative error between model and data,

$$F(\mathbf{x}) = \left\{ \frac{1}{M} \sum_{i=1}^{M} \frac{(d_i - g_i)^2}{d_i^2} \right\}^{1/2}. \tag{5}$$

The problem of finding the minimum of function (5) is usually ill-posed and non-unique.

We may try to improve the situation by reformulating the problem, adding a certain regularization term to expression (5). We thus obtain a new problem, often easier to solve than the initial one.

The second approach is based on the assumption that the Earth's conductivity is a continuous function of depth (Oldenburg, 1979).

For the electromagnetic induction in a one-dimensional structure we have a differential equation:

$$\frac{\mathrm{d}^2 E_x}{\mathrm{d}z^2} + i\omega\mu\sigma E_x = 0. \tag{6}$$

From Equation (6) we can obtain Riccati's equation:

$$\frac{\mathrm{d}R}{\mathrm{d}z} - i\omega R^2 - \mu\sigma(z) = 0, \tag{7}$$

where $R(z, \omega) = B(z, \omega) / E(z, \omega)$ is a transfer function.

A perturbation in conductivity $\delta\sigma$ is followed by a perturbation δT in the transfer function; between these we can write the dependance:

$$\delta R(0, \omega) = \int_0^\infty \mu\sigma(z) \left(\frac{E(z, \omega)}{E(0, \omega)} \right)^2 \delta m(z)\, \mathrm{d}z, \tag{8}$$

where $m(z) = \ln(1/\sigma(z))$.

For practical reasons it is often more convenient to look separately at the phases and amplitudes of the transfer function.

Finally, we can write down expression (8) for each frequency ω_j as follows:

$$\delta T_j = \int_0^\infty G_j(z) \delta m(z)\, \mathrm{d}z, \tag{9}$$

for $j = 1, 2, \ldots, M$, and G_j denotes the amplitude or phase kernel, and T_j the corresponding transfer function.

To construct a model we first choose an initial approximation $m_0(z)$ and compute the transfer functions T_j. We assume δT_j to be the difference between $T_j^0 - T_j$, where T_j^0 is the observed datum.

Solving M equations of the type (9) we find $\delta m(z)$, and the new model $m_1(z) = m_0(z) + \delta m(z)$.

The rate of convergence will be determined by the root-mean-square relative error ϵ,

$$\epsilon = \left(\frac{1}{M} \sum_{j=1}^{M} \left(\frac{\delta T_j}{\delta T_j^0} \right)^2 \right)^{1/2}. \tag{10}$$

We see that in this case we must solve an ill-posed problem, since our aim is to solve Fredholm's integral equation of the first kind.

Structural appraisal of the model is carried out by considering averages of the model at any depth of interest (Backus and Gilbert, 1970).

$$\langle m(z_0) \rangle = \int_0^\infty m(z) A(z, z_0)\, dz, \tag{11}$$

where $A(z, z_0) = \Sigma_{j=1}^{M} a_j(z_0) G_j(z)$ is called the averaging function.

The coefficients $a_j(z_0)$ are a set of constants which are computed to make the averaging function a delta-like function.

The importance of the averaging functions is that they help us to interpret the numerical results. All models which are linearly close to the constructed model and which reproduce the observations have the same average $\langle m(z_0) \rangle$.

Parametric inversion and continuous inversion lead to the solution of systems of linear equations in the least squares sense.

Let us now turn our attention to the algebraic aspects of the numerical solution of the set of linear equations.

$$A\mathbf{x} = \mathbf{b}, \qquad A(MxN), \qquad M \geqslant N. \tag{12}$$

This set of equations often arises from discretization of ill-posed problems and the task of solving such sets of equations in inverse problems is very important.

We are looking for a solution of (12) in the least squares sense, so we have the minimum problem

$$\min_{\mathbf{x}} (\|A\mathbf{x} - \mathbf{b}\|_2). \tag{13}$$

In many methods for ill-posed problems, A in (12) is replaced by a modified matrix $\hat{A}$ such that the difference A-$\hat{A}$ is small and the condition number of $\hat{A}$ is of moderate size.

This procedure is often called the regularization method (Tikhonov, 1963).

The numerical solution of the linear system (12) can be completely analysed in terms of singular value decomposition (SVD) of the matrix A.

The SVD of the matrix A can be expressed as follows:

$$A = U \Sigma V^T = \sum_{i=1}^{N} \sigma_i u V_i^T = v_i^T \tag{14}$$

where $U=(u_1, u_2, ..., u_N)$, $V=(v_1, v_2, ..., v_N)$, $\Sigma=\text{diag}\,(\sigma_1, \sigma_2, ..., \sigma_N)$.

Here U and V are orthogonal matrices and the singular values of matrix A can be assumed to be ordered so that

$$\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_N \geqslant 0. \tag{15}$$

A pseudoinverse solution of (12) can be expressed as follows:

$$\mathbf{x} = \sum_{\sigma_i \neq 0} (\alpha_i/\sigma_i) v_i, \qquad \alpha_i = u_i^T \mathbf{b}. \tag{16}$$

Consequent upon the ill-posedness of the main problem, σ_i approaches zero rapidly for increasing i.

From (16) it can be seen immediatly that small disturbances in the right hand vector $\mathbf{b}$ will result in large changes in the solution $\mathbf{x}$.

Let us now look at two ways of overcoming this difficulty. The first way that suggests itself is to truncate the expression (16) after $k < N$ terms, this is the method of truncated singular value decomposition. In this case we get:

$$\mathbf{x}^{(k)} = \sum_{i=1}^{k} (\alpha_i/\sigma_i) v_i. \tag{17}$$

This procedure is equivalent to solving the modified problem:

$$\hat{A}\mathbf{x}^{(k)} = \mathbf{b}, \tag{18}$$

where $$A - \hat{A} = \sum_{i=k+1}^{N} \sigma_i u_i v_i^T.$$

Efficient numerical algorithms for computing the singular value decomposition were lately given in a work by Dongarra *et al.* (1979).

The second way, which seems to be more flexible than truncated SVD and which has received much attention in recent years, is the method of regularization proposed independantly by Phillips (1962) and Tikhonov (1963).

The minimization problem (13) is replaced by another problem, namely:

$$\min_{\mathbf{x}} \left\{ \|A\mathbf{x} - \mathbf{b}\|_2^2 + \mu \|L\mathbf{x}\|_2^2 \right\}, \tag{19}$$

where the L (pxN) matrix is usually a discrete approximation to some derivative operator and μ is a parameter to be chosen.

For the case $L=I$, where I denotes the unit matrix, the solution to problem (19) can be expressed in the following form (Golub, 1973):

$$\mathbf{x}_\mu = \sum_{i=1}^{N} (\sigma_i/(\sigma_i + \mu^2)) \alpha_i v_i. \tag{20}$$

Thus, the effect of the regularization can be described as applying in the singular value

expansion a filter factor $\sigma_i^2/(\sigma_i^2+\mu^2)$.

Compared to the method of truncated SVD, the method of regularization gives a smoother transition in the damping of components corresponding to small σ_i.

Summing up, both methods expand the solution in the same set of basis vectors v_i, and it can be shown that they often give similar results (Varah, 1979).

Finally, we can notice that even in the case of one-dimensional inverse problems we have to overcome considerable mathematical difficulties. The problem of effective solution of two-dimensional and three-dimensional inverse problems still remains an open question.

4. Summary

Methods used up to now for solving direct problems enable us to analyze two-dimensional models effectively. But current theory on solving three-dimensional problems does not allow us to cope practically with models reflecting real geological situations.

In inverse problems, theory has developed in such a way as to permit its practical application to one-dimensional problems. But even in these cases, because of the ill-posedness of problems, the interpretation of sounding curves, which are obtained from the real experimental data, is usually non-unique. There are two kinds of approaches, the first parametric, and the second based on the assumption of a continuous distribution of electrical conductivity of the Earth. Of these, the second seems to be more general and leads to an algorithm with faster convergence.

Also, as regards two- and three-dimensional inverse problems, despite the theory in two dimensions (Weidelt, 1975c; Jupp and Vozoff, 1977b; Cerv and Pek, 1980), fully effective algorithms of two-dimensional inversion do not exist.

From the practical point of view, when we are handling real observational data, the interpretation of these data is not completely automated. It still depends on the subjective decisions of the interpreter. In interpreting these data, limitations on the number of unknown parameters must be imposed on the basis of the results of other geophysical methods.

References

Backus, G. and Gilbert, F.: 1970, 'Uniqueness in the Inversion of Inaccurate Gross Earth Data', *Phil. Trans. Roy. Soc.* **A266**, 123–192.

Braham, B. *et al.*: 1978, 'Lecture Notes from the U.S.-Australia Electromagnetic Workshop', *Bull. Aust. Soc. Explor. Geophys.* **9**, 2–33.

Brewitt-Taylor, C. R. and Johns, P. B.: 1977, 'Diakoptic Solution of Induction Problems', *IAGA/IAMAP Joint Assembly*, Seattle, August 22 to September 3, 1977.

Cerv, V. and Pek, J.: 1980, 'Numerical Solution of Two-Dimensional Inverse Geomagnetic Induction Problem', *Studia Geoph. et Geod.* **24**, (in press).

Dawson, T. W. and Weaver, J. T.: 1979, 'Three-Dimensional Induction in a Non-Uniform Thin Sheet at the Surface of Uniformly Conducting Earth', *Geophys. J. Roy. Astr. Soc.* **59**, 445–462.

Dey, A.: *Berkeley Modeling and Results,* in Braham *et al.*, (1978).

Dongarra, J. J., Bunch, J. R., Moller, C. B., and Stewart, G. W.: 1979, *LINPACK Users' Guide*, SIAM Publications.

Golub, G. H.: 1973, 'Some Modified Eigenvalue Problems', *SIAM Review* **15**, 318–334.

Hohman, G.: 1978, *3D Modeling,* in Braham *et al.*, (1978).

Jones, F. W. and Vozoff, K.: 1978, 'The Calculation of Magnetotelluric Quantities for Three-Dimensional Conductivity Inhomogeneities', *Geophysics* **43**, 1167–1175.

Jupp, D. L. B. and Vozoff, K.: 1975a, 'Stable Interative Methods for Inversion of Geophysical Data', *Geophys. J. Roy. Astr. Soc.* **42**, 957–976.

Jupp, D. L. B. and Vozoff, K.: 1977b, 'Two-Dimensional Magnetotelluric Inversion', *Geophys J. Roy. Astr. Soc.*, **50**, 333–352.

Oldenburg, D. W.: 1979, 'One-Dimensional Inversion of Natural Source Magnetotelluric Observations', *Geophysics* **44**, 1218–1244.

Phillips, D. L.: 1962, 'A Technique for Numerical Solution of Certain Integral Equations of the First Kind', *J. ACM* **9**, 84–97.

Pridmore, D. F.: 1978, 'Three-Dimensional Modeling of Electric and Electromagnetic Data Using Finite Element Method', Ph. D. Thesis, Univ. of Utah, 249 pp.

Tarlowski, Z.: 1978, 'Application of the Finite Difference Method to Explain the Electromagnetic Induction in Two-Dimensional Structures', Ph. D. Thesis, Institute of Geophysics of Polish Academy of Sciences.

Tikhonov, A. N.: 1963, 'Solution of Incorrectly Formulated Problems and the Regularization Method', *Soviet Math. Dokl.* **4**, 1035–1038.

Varah, J. M.: 1979, 'A Practical Examination of Some Mumerical Methods for Linear Discrete Ill-posed Problems', *SIAM Review* **21**, 100–111.

Varga, R.S.: 1962, *Matrix Iterative Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 322 pp.

Weaver, J. T.: 1979, 'Electromagnetic Induction in Thin Sheet Conductivity Anomalies at the Surface of the Earth', *Proc. IEEE* **67**, 1044–1050.

Weaver, J. T. and Brewitt-Taylor, C. R.: 1978, 'Improved Boundary Conditions for the Numerical Solution of E-Polarization Problems in Geomagnetic Induction', *Geophys. J. Roy Astr. Soc.* **54**, 309–317.

Weidelt, P.: 1972a, 'The Inversion of Geomagnetic Induction', *Zeitschrift für Geoph.* **38**, 257–289.

Weidelt, P.: 1975b, 'Electromagnetic Induction in Three-Dimensional Structures', *J. Geophys.* **41**, 89–109.

Weidelt, P.: 1975c, 'Inversion of Two-Dimensional Conductivity Structures', *Phys. Earth Plant. Int.* **10**, 282–291.

Wieladek, R., Nowozynski, K., and Tarlowski, Z.: 1980, 'Application of the Cholesky-Banachiewicz Method to Solving Some Linear Systems of Equations Approximating Helmholtz's Equation', Publications of the Institute of Geophysics Polish Academy of Sciences (in press).