Physics of the Earth and Planetary Interiors, 10 (1975) 282–291

INVERSION OF TWO-DIMENSIONAL CONDUCTIVITY STRUCTURES

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(Accepted for publication February 27, 1975)

This review shows that the inversion of two-dimensional structures is still an almost uninvestigated area. Only for very restricted anomalous domains exact methods exist: (a) undulated interface between perfect conductor and insulator; (b) thin non-uniform sheet. So far, real conductors must be inverted by linearization. A method for the computation of the pertinent kernels is described and the well-developed method of generalized matrix inversion is applied in a preliminary study both to artificial and real data.

1. Introduction

Upon the determination of the change of normal conductivity with depth by a one-dimensional inverse problem, attention is drawn to the next complicated inverse problem: Assume that within a known normal conductivity structure there is embedded an unknown, laterally non-uniform, anomalous domain with constant cross-section in the x-direction, and of limited extent both in the other horizontal direction and in depth (y- and z-direction, respectively). The additional assumption of an x-independent inducing magnetic field renders the configuration purely two-dimensional. Then the problem to be solved is to deduce the conductivity within the anomalous domain from a knowledge of the normal conductivity structure and the anomalous electromagnetic field, observed for various frequencies at the surface of the earth. Since more a local then a global feature is considered, the assumption of a plane earth is justified.

For a two-dimensional configuration, Maxwell's equations are split into two disjoint sets, named according to the component in the x-direction either E-polarization (E_x, H_y, H_z), or H-polarization (H_x, E_y, E_z). For simplicity, the following considerations are confined to the more interesting E-polarization case, where both the electric and magnetic surface field are disturbed. Using cartesian coordinates x, y, z with z positive downwards, SI units and a time factor $e^{i\omega t}$ throughout, the pertinent equations are:

$$i\omega\mu_0 H_y = -\frac{\partial E_x}{\partial z}, \qquad i\omega\mu_0 H_z = \frac{\partial E_x}{\partial y} \tag{1a,b}$$

$$\sigma E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \tag{2}$$

leading to the differential equation:

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = i\omega\mu_0 \sigma E_x \tag{3}$$

In the E-polarization case the inducing field might be due to any two-dimensional current distribution, but the subsequent considerations are simplified, if a quasi-uniform external magnetic field is assumed.

The interpretation is based on one or more of the transfer functions:

$$E_{xa}/E_{xn}, \qquad H_{ya}/H_{yn}, \qquad H_z/H_{yn} \tag{4}$$

which are functions of frequency ω and space coordinate y. The subscripts n and a refer to the normal and anomalous part of the respective field quantity.

It is generally accepted that a perfect knowledge of any one of the transfer functions (4) contains sufficient information to reveal the anomalous conductivity in a unique way; the dependence on y provides the lateral resolution and the dependence on ω gives the resolution with depth. So far, however, a proof of this assertion has not been given. Apart from the question of uniqueness, there remains also unanswered the question of existence of a solution; i.e., which conditions a data set has to satisfy to belong to an anomalous conductivity structure. From the answer, certain compatibility relations can be derived (e.g. statements on the smoothness of the frequency dependence), which experimental data have to satisfy to be exactly interpretable. Although it is expected from experience with the one-dimensional inverse problem that a uniqueness proof will not offer a very appealing way to find the conductivity in practice (Bailey, 1973), it will consolidate the fundaments of induction theory.

Apart from two degenerate cases, which will be considered later on, there appears to exist no specific approach to invert two-dimensional structures. In general, starting with a simple, plausible model, trial and error techniques are applied. In this way, an excellent agreement is sometimes obtained between observations and model interpretations for various frequencies. Work along these lines has been carried out by Schmucker (1964, 1970), Filloux (1967), Swift (1967), Cochrane and Hyndman (1970), Bennett and Lilley (1971), Bennett (1972), Greenhouse (1972), Hyndman and Cochrane (1971), Scheelke (1972), Dragert (1973), Steveling (1973), and Winter (1973).

Only in the two complementary cases where either the anomaly is due to an undulation of the deep interface between an insulator and a perfect conductor, or the anomaly is due to lateral conductivity variation in a thin-surface anomaly, exact inverse methods exist. These will be considered in Section 2. So far, the only way to handle the inversion in finite conductors is to linearize the problem and to apply the tools of generalized linear inversion. This is worked out in Section 3.

2. Degenerate cases

2.1. Undulation of a perfectly conducting interface

When the conductor can be approximated by an insulator with a perfectly conducting substratum, the interface is a magnetic line of force, since no magnetic component normal to the interface exists. Hence, the family of field lines, which give rise to the surface field have to be computed. This can be done by searching for a set of current sources beneath the surface $z = 0$, which can account for the observed anomalous field (Siebert, 1974). The differential equation for a magnetic line of force is $H_y/H_z = \mathrm{d}y/\mathrm{d}z$, or introducing the vector potential A_x with $\mu_0 H_y = \partial A_x/\partial z$, $\mu_0 H_z = -\partial A_x/\partial y$ it yields $(\partial A_x/\partial y)\mathrm{d}y + (\partial A_x/\partial z)\mathrm{d}z = 0$. Hence, the lines A_x = constant are the field lines. It is assumed that the field is undisturbed for $|y| \to \infty$. Since the vector potential of a single-line current is proportional to $\log |y|$ for $|y| \to \infty$, a representation of the field in terms of current dipoles and higher multi-

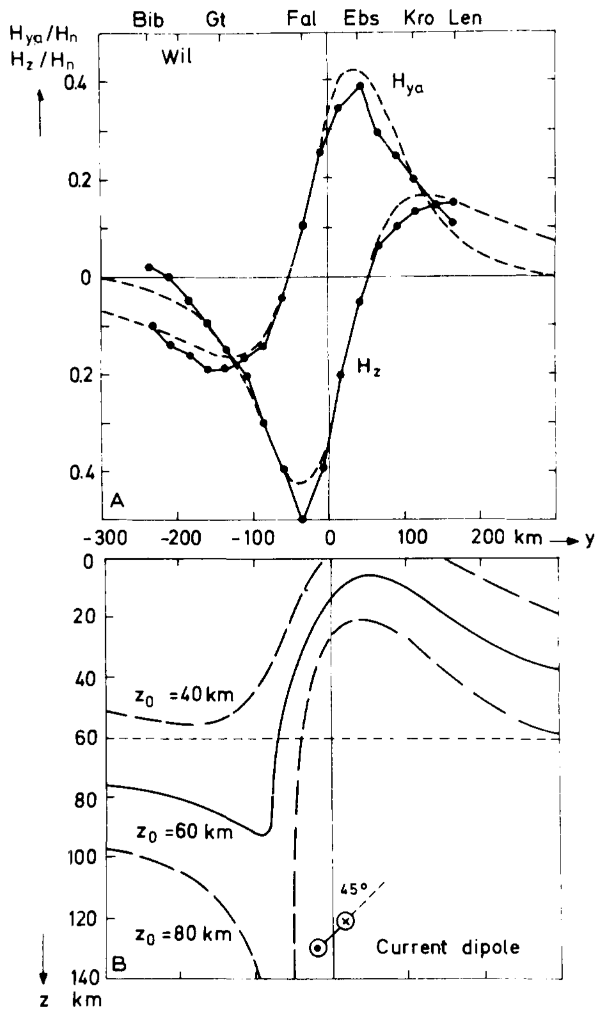

Fig. 1 A. Interpretation of the anomalous field of the north German conductivity anomaly (full lines) by a current dipole (dashed lines).
B. The family of equivalent undulations of a perfect conductor (after Siebert, 1974).

poles is required. The vector potential of a current dipole at r_k is:

$$A_{xk}(r) = D_k \cdot (r - r_k)/|r - r_k|^2$$

where the vector D_k points from the current in $-x$-direction to the current in $+x$-direction. The vector potential of H_{yn} is $\mu_0 H_{yn} z$, and considering a field line with $z \to z_0$ for $|y| \to \infty$, for a representation in terms of dipoles the implicit equation:

$$(z - z_0) + \sum_k \tilde{D}_k \cdot (r - r_k)/|r - r_k|^2 = 0 \qquad (5)$$

$$\tilde{D}_k = D_k/(\mu_0 H_{yn})$$

is obtained, which for fixed y and z_0 might be solved by Wegstein iteration, starting with $z = z_0$. It results a family of curves with z_0 as parameter. A proper z_0 must be chosen by physical reasoning, in particular $\min\{z(y)\} > 0$ has to be satisfied. Siebert (1974) interprets the north German conductivity anomaly (Schmucker, 1959) by a single current dipole ($y_1 = 0$, $z_1 = 127$ km, $|\tilde{D}_1| = 7{,}600$ km^2, inclination 45°). The result is shown in Fig. 1. None of the three curves appears to be very realistic: The interface $z_0 = 40$ km cuts the surface, the interface $z_0 = 80$ km is pretty steep, and the most reasonable interface $z_0 = 60$ km approaches the surface up to 7 km.

2.2. Inversion of thin sheets

Assume that the anomaly results from a lateral variation of the integrated conductivity τ in a thin surface sheet at $z = 0$ with known horizontal layering for $z > 0$. Let:

$$\tau(y) = \tau_n + \tau_a(y)\ , \qquad \tau_a(y) = 0 \qquad \text{for } |y| \to \infty$$

where τ_n is known and $\tau_a(y)$ is to be determined. For the E-polarization case Schmucker (1971a,b) proposes the following method of solution.

From the sheet-current density:

$$\tau E_x = H_y^+ - H_y^- = (H_{yn}^+ - H_{yn}^-) + (H_{ya}^+ - H_{ya}^-)$$

$$= \tau_n E_{xn} + H_{ya}^+ - H_{ya}^-$$

follows:

$$\tau_a = \frac{H_{ya}^+ - H_{ya}^- - \tau_n E_{xa}}{E_x} \qquad (6)$$

where the superscripts + and − refer to the upper ($z = -0$) and lower side ($z = +0$) of the sheet, respectively. The vertical component H_z is continuous across the sheet. There exist two kernels K^+ and K^-, which admit a convolution integral representation of $H_{ya}^{\pm}$ in terms of H_z:

$$H_{ya}^{\pm}(y) = \int_{-\infty}^{+\infty} K^{\pm}(y - \eta) H_z(\eta) \mathrm{d}\eta \mathrel{\hat=} K^{\pm} \ast H_z \qquad (7)$$

K^+ and K^- depend only on the conductivity above and below the sheet, respectively. Since a surface sheet is assumed, K^+ is in fact independent of conductivity and is given by:

$$K^+(y) = -\frac{1}{\pi y}$$

i.e. the negative of the familiar K-operator (Siebert and Kertz, 1957). Examples for K^- are:

(a) For zero conductivity in $0 < z < h$ and a perfect conductor at $z = h$:

$$K^-(y) = \frac{1}{2h} \coth\left(\frac{\pi y}{2h}\right)$$

(b) For a uniform halfspace with conductivity σ_0:

$$K^-(y) = \frac{k}{\pi}\left\{\frac{\pi}{2} + \int_{k|y|}^{\infty} K_1(u) \frac{\mathrm{d}u}{u}\right\} \cdot \mathrm{sgn}(y)$$

where K_1 is a modified Bessel function of the second kind and $k^2 = \mathrm{i}\omega\mu_0\sigma_0$.

From eqs. 6, 7, and 1b follows:

$$\tau_a = \frac{(K^+ - K^-) \ast \partial E_{xa}/\partial y - \mathrm{i}\omega\mu_0\tau_n E_{xa}}{\mathrm{i}\omega\mu_0(E_{xn} + E_{xa})} \qquad (8)$$

Thus $\tau_a(y)$ can be determined from a knowledge of the transfer function E_{xa}/E_{xn} for one frequency only. If instead the transfer function $z_H = H_z/H_{yn}$ is given, eq. 8 reads:

$$\tau_a = \frac{(K^+ - K^-) \ast H_z - \mathrm{i}\omega\mu_0\tau_n G \ast H_z}{C^+ H_{yn} + G \ast H_z} \qquad (9)$$

where:

$$E_{xa} = \mathrm{i}\omega\mu_0 G \ast H_z$$

$$G = \tfrac{1}{2}\mathrm{sgn}(y)$$

and

$E_{xn} = i\omega\mu_0 C^+ H^+_{yn}$

have been applied. C^+ depends on the normal conductivity structure and yields for the two cases mentioned above:

(a) $C^+ = h/(1 + i\omega\mu_0\tau_n h)$

(b) $C^+ = 1/(k + i\omega\mu_0\tau_n)$

Details for the computation of all kernels are given by Schmucker (1969, 1971a,b). If the data are perfect, and the normal conductivity is completely known, the same conductivity profile $\tau(y)$ will be obtained for all frequencies. Moreover, this quantity will be real. In practice the normal conductivity structure must be varied in order to minimize the quadrature part of $\tau_a(y)$ and to maximize the agreement between the conductivity profiles for all frequencies. A successful application of this inversion procedure is given by Schmucker (1971a).

3. Inversion of thick conductors by linearization

3.1. Computation of the partial derivatives

The inverse problem of electromagnetic induction is a non-linear problem. The most popular way to handle non-linear problems is to linearize them. Assume that the anomalous domain consists of M cells of known size and constant unknown conductivity σ_j, $j = 1, ..., M$. For conciseness let $\boldsymbol{\sigma}^T = (\sigma_1, ..., \sigma_M)$. Associated with each datum is a function f_i, which transforms σ into the datum g_i:

$$g_i = f_i(\sigma), \qquad i = 1, ..., N \tag{10}$$

Suppose that an approximation $\boldsymbol{\sigma}^0$ to $\boldsymbol{\sigma}$ is known. Neglecting terms of order $0(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)^2$, eq. 10 reads:

$$\sum_{j=1}^{M} \frac{\partial f_i}{\partial \sigma_j^0}(\sigma_j - \sigma_j^0) = g_i - f_i(\boldsymbol{\sigma}^0), \qquad i = 1, ..., N \tag{11}$$

If $\boldsymbol{\sigma}^0$ is near $\boldsymbol{\sigma}$, the system (11) with N equations and M unknowns yields a correction to $\boldsymbol{\sigma}^0$, thus starting an iterative scheme. The inversion of system 11 is best carried out by generalized matrix inversion (Section 3.2.). First the partial derivatives $\partial f_i/\partial\sigma_j$ have to be determined. For the moment consider partial derivatives of the electric field only and let E_α and E_β be two solutions of eq. 3 with $\sigma = \sigma_\alpha$ and $\sigma = \sigma_\beta$:

$$\begin{aligned} \Delta E_\alpha(r) &= i\omega\mu_0\sigma_\alpha(r)E_\alpha(r) \\ \Delta E_\beta(r) &= i\omega\mu_0\sigma_\beta(r)E_\beta(r) \end{aligned} \tag{12a,b}$$

(For shortness, the subscript x of E is dropped in the sequal.) The difference (12a)–(12b) is:

$$\Delta(E_\alpha - E_\beta) = i\omega\mu_0\sigma_\beta(E_\alpha - E_\beta) + i\omega\mu_0(\sigma_\alpha - \sigma_\beta)E_\alpha \tag{13}$$

Let $G_\beta(\boldsymbol{r}'|\boldsymbol{r})$ be that solution of:

$$\Delta G_\beta(\boldsymbol{r}'|\boldsymbol{r}) = i\omega\mu_0\sigma_\beta(\boldsymbol{r})G_\beta(\boldsymbol{r}'|\boldsymbol{r}) - \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{14}$$

which vanishes at infinity. Multiply eq. 14 by $E_\alpha(\boldsymbol{r}') - E_\beta(\boldsymbol{r}')$ and eq. 13 by $G_\beta(\boldsymbol{r}'|\boldsymbol{r})$ and integrate the difference with respect to $\boldsymbol{r}'$ over the whole (y,z)-plane. Then Green's theorem yields:

$$E_\alpha(\boldsymbol{r}) - E_\beta(\boldsymbol{r}) = -i\omega\mu_0 \int \{\sigma_\alpha(\boldsymbol{r}') - \sigma_\beta(\boldsymbol{r}')\} \times E_\alpha(\boldsymbol{r}')G_\beta(\boldsymbol{r}'|\boldsymbol{r})\mathrm{d}A' \tag{15}$$

The domain of integration is within the (y,z)-plane the region where $\sigma_\alpha \neq \sigma_\beta$. If $\sigma_\alpha = \sigma$, $\sigma_\beta = \sigma_n$, eq. 15 reads:

$$E(\boldsymbol{r}) = E_n(\boldsymbol{r}) - i\omega\mu_0 \int \sigma_a(\boldsymbol{r}')E(\boldsymbol{r}')G_n(\boldsymbol{r}'|\boldsymbol{r})\mathrm{d}A' \tag{16}$$

where $\sigma_a = \sigma - \sigma_n$.

This is an efficient integral equation for E, provided that G_n is known (Hohmann, 1971). Formulae for G_n are given below. For an infinitesimal conductivity difference $\delta\sigma = \sigma_\alpha - \sigma_\beta$, eq. 15 yields:

$$\delta E(\boldsymbol{r}) = -i\omega\mu_0 \int \delta\sigma(\boldsymbol{r}')E(\boldsymbol{r}')G(\boldsymbol{r}'|\boldsymbol{r})\mathrm{d}A'$$

where E and G correspond to $\sigma = \sigma_\alpha \approx \sigma_\beta$.

In the case of cells with constant conductivity assuming that E is constant within each cell this leads to:

$$\begin{aligned} \frac{\partial E(\boldsymbol{r}_i)}{\partial \sigma_j} &= -i\omega\mu_0 E(\boldsymbol{r}_j)\Gamma_{ji} \\ \Gamma_{ji} &= \int_{c_j} G(\boldsymbol{r}'|\boldsymbol{r}_i)\mathrm{d}A' \end{aligned} \tag{17a,b}$$

where c_j is the cell centered at $\boldsymbol{r}_j$. Hence, via eqs. 17a,b the partial derivatives are closely related to the Green's function $G(\boldsymbol{r}'|\boldsymbol{r})$, which satisfies the integral equation:

$$G(\boldsymbol{r}'|\boldsymbol{r}) = G_n(\boldsymbol{r}'|\boldsymbol{r}) - i\omega\mu_0 \int \sigma_a(\boldsymbol{r}'')G_n(\boldsymbol{r}'|\boldsymbol{r}'')G(\boldsymbol{r}''|\boldsymbol{r})\mathrm{d}A'' \tag{18}$$

derived along the lines of eqs. 12–16. Integrating eq. 18 with respect to $\boldsymbol{r}'$ and using eq. 17b yields:

$$\Gamma_{ji} = \Gamma_{nji} - i\omega\mu_0 \sum_{k=1}^{M} \sigma_{ak}\Gamma_{njk}\Gamma_{ki} \ , \qquad i = 1, ..., N_E(\omega), \qquad j = 1, ..., M \tag{19}$$

$N_E(\omega)$ is the number of points, for which surface values of E at the frequency ω are given. For i fixed 19 is a system of M equations for the M unknowns Γ_{ji}. The solution is simplified by the dominant diagonal due to the logarithmic singularity of $G_n(\boldsymbol{r}'|\boldsymbol{r})$ for $\boldsymbol{r} \to \boldsymbol{r}'$. The partial derivatives for the magnetic field components are obtained similarly. According whether H_y or H_z is considered, eqs. 17a,b and 18 are differentiated with respect to z or y (coordinates of the point of observation).

It remains to determine $G_n(\boldsymbol{r}'|\boldsymbol{r})$, which can be conceived as the electric field of a unit line current placed at $\boldsymbol{r}'$ and observed at $\boldsymbol{r}$. The reciprocity relation for Green's functions requires $G_n(\boldsymbol{r}'|\boldsymbol{r}) = G_n(\boldsymbol{r}|\boldsymbol{r}')$. The normal conductivity structure consists in $z > 0$ of L layers with conductivities σ_{nm}, $m = 1, ..., L$ and interfaces at $h_1 = 0, h_2, ..., h_L$ and in $z \leqslant 0$ of a nonconducting air half-space ($\sigma_{n0} = 0$). Required is the solution of:

$$\Delta G_n(\boldsymbol{r}'|\boldsymbol{r}) = i\omega\mu_0\sigma_n(\boldsymbol{r})G_n(\boldsymbol{r}'|\boldsymbol{r}) - \delta(\boldsymbol{r} - \boldsymbol{r}')$$

which vanishes at infinity. Let the source and observation point be placed in the μth and mth layer, respectively, and let in the mth layer:

$$G_n^m(\boldsymbol{r}'|\boldsymbol{r}) = \int_0^\infty \{P_m^+ + P_m^-\}\cos\lambda(y - y')\mathrm{d}\lambda \tag{20}$$

where:

$$P_m^\pm = \begin{cases} \gamma_0 A_m^\pm f_m^\pm(z), & z \leqslant z' \\ \gamma_L B_m^\pm f_m^\pm(z), & z \geqslant z' \end{cases}$$

$$f_m^\pm(z) = \exp\{\pm\alpha_m(z - h_m)\} \qquad \alpha_m^2 = \lambda^2 + i\omega\mu_0\sigma_{nm}$$

γ_0 and γ_L can be so adjusted that $A_0^+ = B_L^- = 1$. Since there are no sources in $z \leqslant 0$ and for $z \geqslant z'$ if z' is in the Lth layer, $A_0^- = B_L^+ = 0$. With these starting values, the continuity of G_n and $\partial G_n/\partial z$ across the interfaces yields the forward and backward recurrence relations:

$$A_m^\pm = (1 \pm \alpha_{m-1}/\alpha_m)g_{m-1}^+A_{m-1}^+ + (1 \mp \alpha_{m-1}/\alpha_m) \times g_{m-1}^-A_{m-1}^- \ , \qquad m = 1, ..., \mu$$

$$B_m^\pm = (1 \pm \alpha_{m+1}/\alpha_m)g_m^\mp B_{m+1}^+ + (1 \mp \alpha_{m+1}/\alpha_m) \times g_m^\mp B_{m+1}^- \ , \qquad m = L-1, ..., \mu$$

with:

$$g_0^\pm = 1/2, \quad g_m^\pm = (1/2)\exp\{\pm\alpha_m(h_{m+1} - h_m)\} \ , \qquad m = 1, ..., L-1$$

In the case $\mu = L$, no recurrence is required for the B terms. The coefficients γ_0 and γ_L are determined from the fact that in eq. 20 the difference in the upward- (downward-)travelling waves for $z > z'$ and $z < z'$ must be due to the primary excitation given by:

$$\frac{1}{2\pi}K_0(k_\mu|\boldsymbol{r} - \boldsymbol{r}'|) = \frac{1}{2\pi}\int_0^\infty e^{-\alpha_\mu|z-z'|}\cos\lambda(y - y')\frac{\mathrm{d}\lambda}{\alpha_\mu}$$

where K_0 is the zero-order modified Bessel function of the second kind and $k_\mu^2 = i\omega\mu_0\sigma_{n\mu}$. Hence:

$$\gamma_0 = \frac{1}{2\pi\alpha_\mu}\cdot\frac{B_\mu^- f_\mu^- + B_\mu^+ f_\mu^+}{A_\mu^+ B_\mu^- - A_\mu^- B_\mu^+}$$

$$\gamma_L = \frac{1}{2\pi\alpha_\mu}\cdot\frac{A_\mu^- f_\mu^- + A_\mu^+ f_\mu^+}{A_\mu^+ B_\mu^- - A_\mu^- B_\mu^+}$$

where $f_\mu^\pm = f_\mu^\pm(z')$.

The nominator (including α_μ) can be considered as the Wronskian of two solutions of:

$$w''(z) = \{\lambda^2 + i\omega\mu_0\sigma_n(\tau)\}\omega(\tau)$$

It is independent of z (and μ), thus ensuring reciprocity. The integration (17b) for a rectangular cell with dimensions L_y and L_z is easily performed by adding in eq. 20 the factor:

$$4\sin(\lambda L_y/2)\sinh(\alpha_\mu L_z/2)/(\lambda\alpha_\mu)$$

and using an obvious modification for the cell where $z = z'$.

An alternative way to compute the partial derivatives is to determine numerically the effect due to a conductivity change in the jth cell. From numerical experiments in the one-dimensional case (Glenn et al., 1973) it is expected that this approach is significantly less accurate.

3.2. Generalized matrix inversion

The system of linear equations arising from the linearization of the inverse problem is best solved by generalized matrix inversion. Excellent discussions of this topic are given by Lanczos (1961), Jackson (1972) and Wiggins (1972). Applications to electromagnetic and geoelectric sounding can be found in Glenn et al. (1973) and Inman et al. (1973). The linearization equation (11) yields N equations for the M unknown parameter corrections written as matrix equation $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{y}'$. Here $\boldsymbol{A}'$ is the $N \times M$ matrix of the partial derivatives, $\boldsymbol{x} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^0$ the parameter correction vector, and $\boldsymbol{y}'$ the N component residuum vector between the data and previous model outcome.

If the errors of the data are known, it is advantageous to transform the equation in such a way that each datum has the same variance σ_0^2. Assuming uncorrelated data, this is easily done by multiplying the jth row by:

$$\sigma_0/\sqrt{\text{var}\,(y_j')}\ , \qquad j = 1, ..., N$$

Thus in a least squares solution, each residuum is weighted with the inverse square root of the variance. In the transformed system:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

the matrix $\boldsymbol{A}$ is now decomposed into eigenvectors:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^T$$

Here $\boldsymbol{V}$ is an $M \times P$ matrix containing the P eigenvectors belonging to the P non-zero eigenvalues of the problem:

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{v}_i = \lambda_i^2 \boldsymbol{v}_i\ , \qquad i = 1, ..., M$$

Similarly, $\boldsymbol{U}$ is an $N \times P$ matrix with the P eigenvectors of the problem:

$$\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{u}_j = \lambda_j^2 \boldsymbol{u}_j\ , \qquad j = 1, ..., N$$

associated with non-zero eigenvalues. $\boldsymbol{\Lambda}$ is a $P \times P$ diagonal matrix with the P non-zero eigenvalues. The generalized inverse of $\boldsymbol{A}$ is:

$$\boldsymbol{H} = \boldsymbol{V}\boldsymbol{\Lambda}^{-1}\boldsymbol{U}^T$$

In the well-posed case $\boldsymbol{H}$ is the ordinary solution ($M = N = P$), in the overconstrained case (i.e. $N > P = M$) $\boldsymbol{H}$ provides a least-square solution, and in the underdetermined case ($N = P < M$) the shortest vector compatible with the data is found. Due to the above transformation, the variance of the component x_j is simply:

$$\text{var}\,(x_j) = \sigma_0^2 \sum_{k=1}^{P} \left(\frac{V_{jk}}{\lambda_k}\right)^2 \qquad (21)$$

From eq. 21 it is seen that the variance is largely due to the small eigenvalues, which should be discarded when a small error is intended. If, however, the number of eigenvectors used to construct the inverse $\boldsymbol{H}$ decreases, the solution degrades, the parameter changes x_j become less resolved, $\boldsymbol{HA}$ will deviate more from an M-element unit matrix than before. The resolved vector $\langle \boldsymbol{x} \rangle$ is related to the true, but unknown vector $\boldsymbol{x}$ through:

$$\langle \boldsymbol{x} \rangle = \boldsymbol{H}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$$

$\boldsymbol{R}$ is named the resolution matrix and is given by:

$$\boldsymbol{R} = \boldsymbol{V}\boldsymbol{\Lambda}^{-1}\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{V}^T \qquad (22)$$

Hence, there is the same trade-off between resolution and error of estimate as is well-known from the Backus-Gilbert theory (Backus and Gilbert, 1970).

Generalized matrix inversion is used both to invert a structure and to estimate the information contents of a given data set. Inverting a structure one has two tools to stabilize the notably unstable inversion scheme: to diminish the number of eigenvectors and to prescribe an upper bound for the parameter changes, e.g. 25% of the actual value of the parameter, leading to a trade-off between convergence rate and stability (Glenn et al., 1973). This has the additional advantage that the searched quantities do not change their sign.

In a first application of generalized matrix inversion, the information contents of different data sets are estimated. A particular resistivity structure, shown in the top of Fig. 2, is assumed, and for this structure the pertinent surface data and kernels for eleven different combinations of periods and components are computed for eleven points at the surface. Further it is assumed that the in-phase and quadrature part of each datum have an error of 10% of the modulus of the datum. Then the resolution matrix has been computed under the assumption that the error for each x_j should be near to 20%, thus determining according to (21) the number of retained eigenvectors for each row of (22) separately. The 6th row of the 32 X 32 resolution matrix $\boldsymbol{R}$ is shown in Fig. 2 for the different data cases. For perfect resolution the 6th component of this row would be unity, the remaining components zero. The cell, where the resolution is maximized is marked by a black arrow head. The other arrows were only drawn when their length was longer than one half of the length of an arrow head. The white arrows show the relative weights, with which the other cells enter in an estimate of cell 6.

For each cell an averaging cross-section q_j in units of the area of a cell has been determined according to:

$$q_j = r_{jj}^{-1} \sum_{k=1}^{M} |r_{jk}| \tag{23}$$

where r_{jk} are the elements of $\boldsymbol{R}$. This number is given below each small figure of Fig. 2. In the present example the estimation of the data contents started with the assumed model; in applications the last iteration is the appropriate starting point.

Numerical experiments were performed to test the capability of generalized matrix inversion for the inversion of two-dimensional structures. In one example, for the four periods mentioned in Fig. 2 the H_z compo-

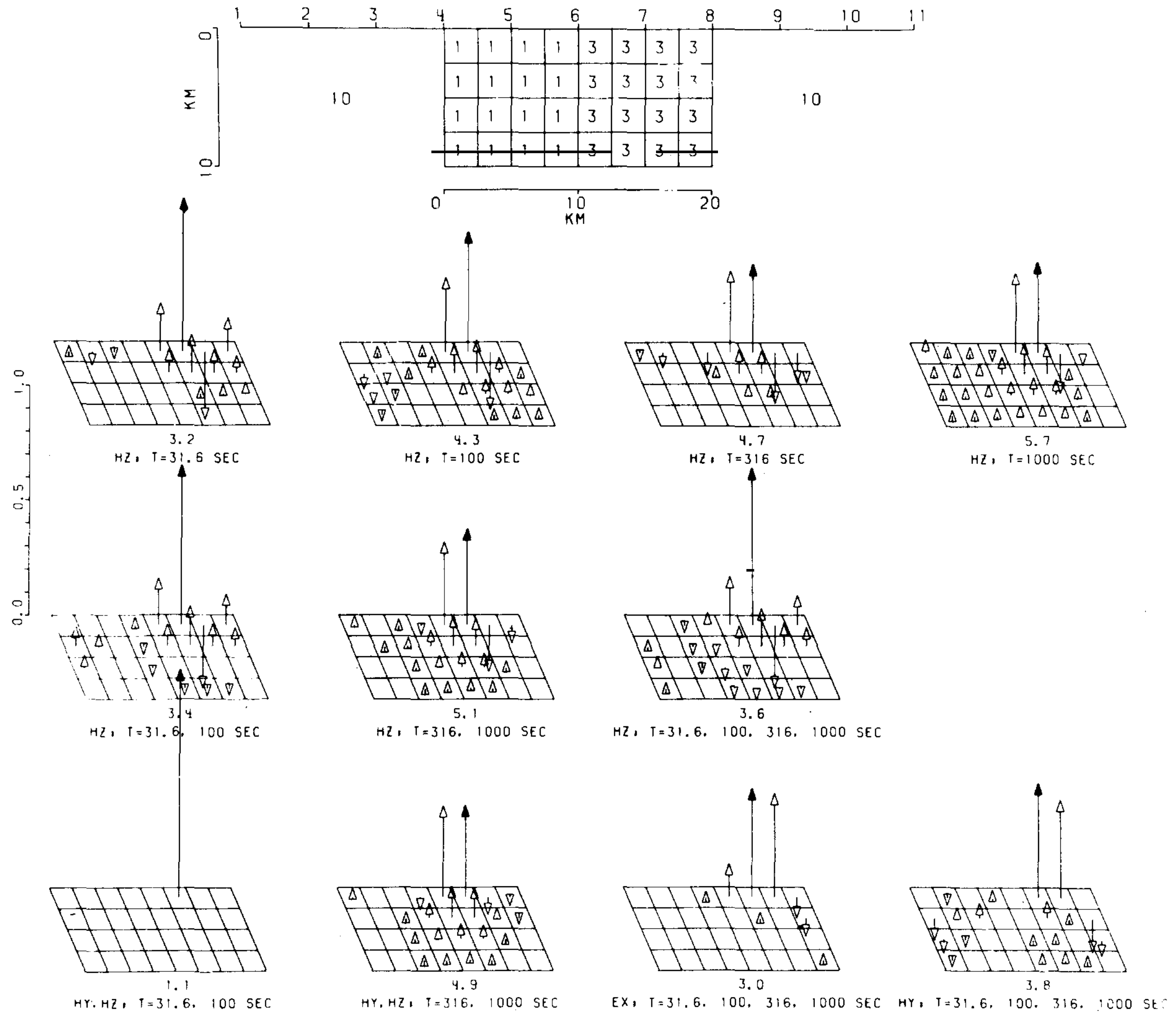

Fig. 2. Example for the estimation of the information contents of a given data set for a particular resistivity structure (resistivities in Ωm). Resolution function for the 6th cell under various conditions (full explanation in the text).

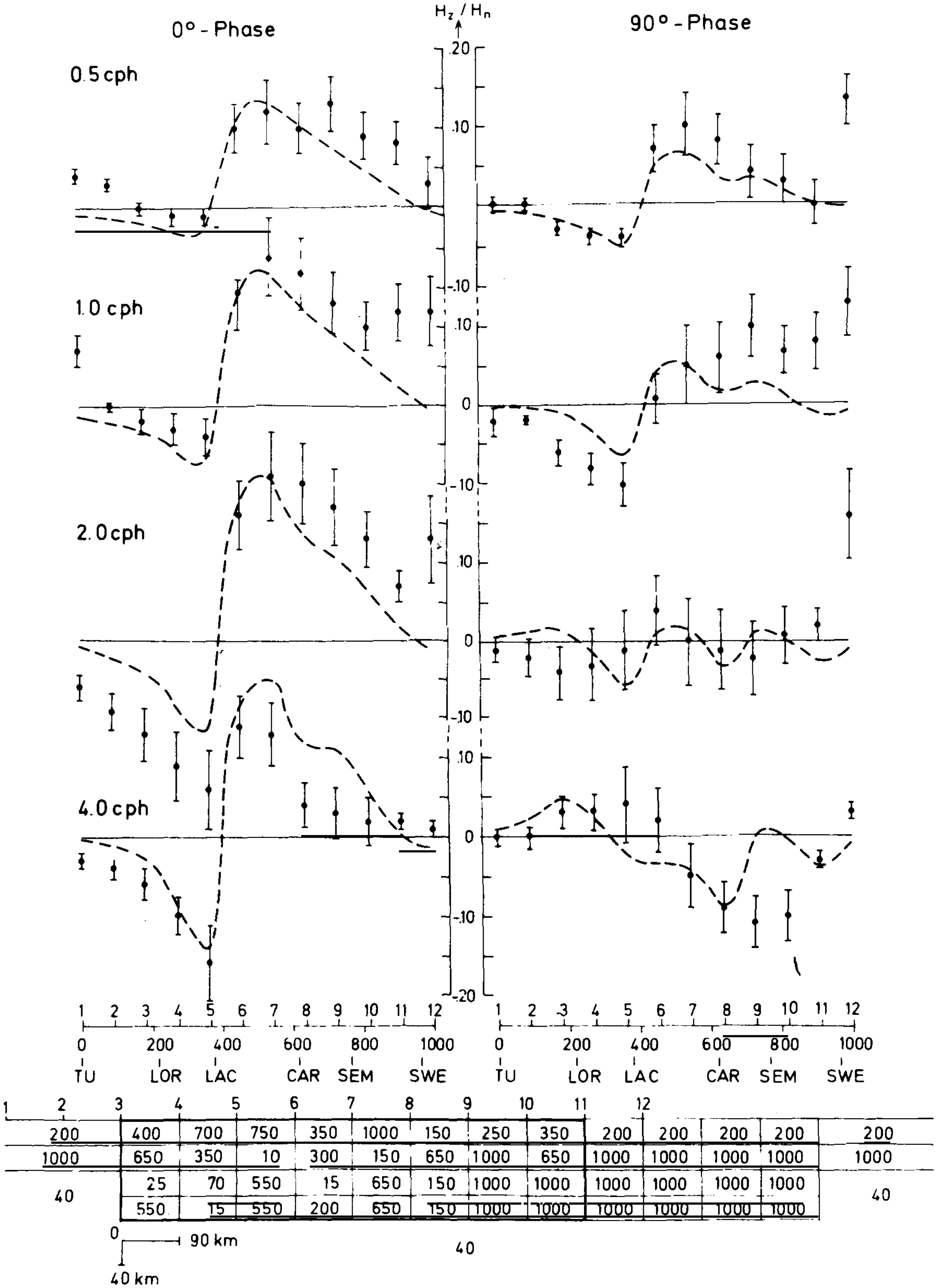

Fig. 3. Interpretation of the Rio Grande anomaly. Only the left 32 cells were allowed to change. Given is H_z for four frequencies. The dashed lines refer to the model given at the bottom (resistivities in Ωm).

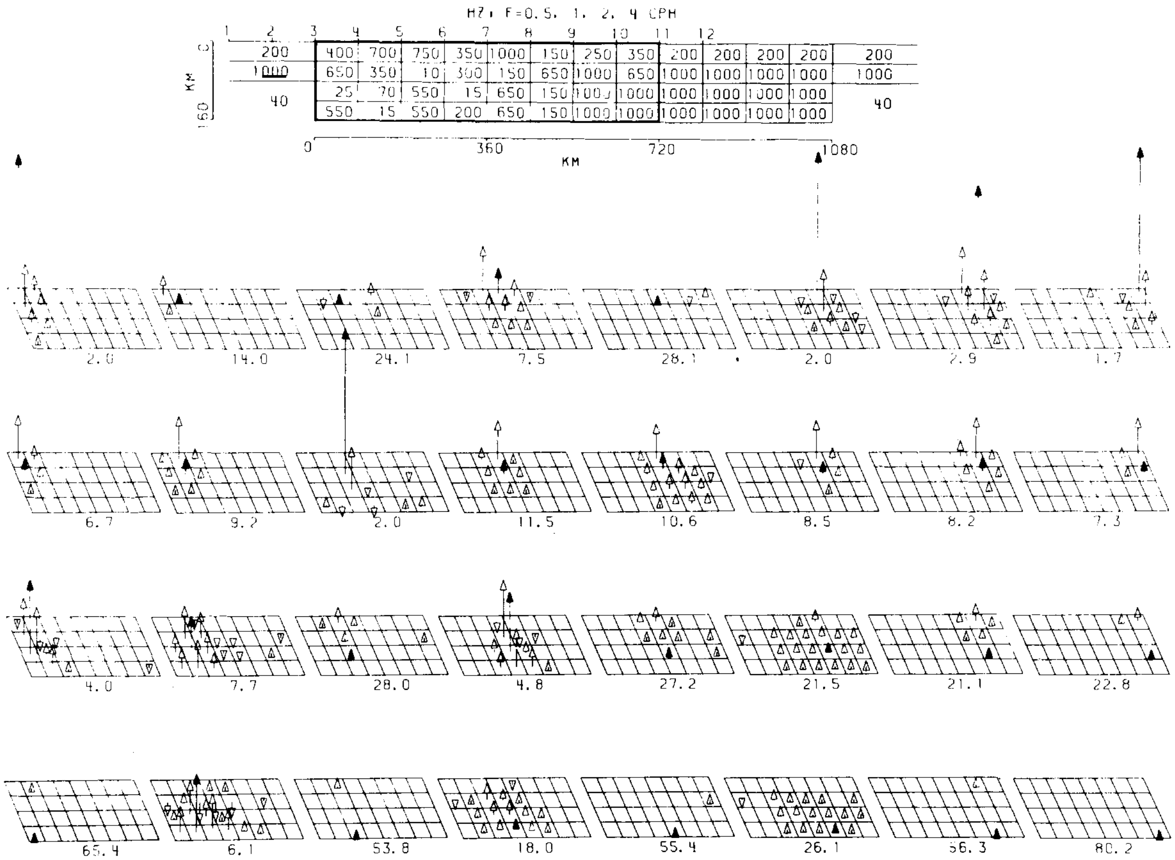

Fig. 4. The complete resolution matrix for the Rio Grande anomaly using the result of Fig. 3 and admitting an error of 20% for the averaged conductivities.

nent at the same surface points and for the same resistivity structure as in Fig. 2 were used as input data. Using the normal resistivity structure as zero-order approximation, the iterative scheme converged to the correct values, provided that the anomalous domain was decomposed into 8 cells with 5-km edges, whereas this first approximation was diverging for 32 cells with 2.5-km edges. This suggests the strategy of starting the inversion with a coarse grid, which is refined later on.

An application of the inversion procedure to the Rio Grande anomaly (Schmucker, 1970) is given in Fig. 3. The data set consists of the H_z values for four periods over a profile of 990 km, interpolated for 12 equidistant data points. The anomalous domain consists of 48 cells, 90 × 40 km^2. During the iteration only the resistivity in 32 cells was allowed to change. Using the normal resistivity structure given in Fig. 3, and as initial guess 200 Ωm for the first row of the anomalous domain and 1,000 Ωm for the other, the RMS error of fit between data and model outcome first decreased and then divergence occurred. Fig. 3 shows the model for the least RMS error obtained.

Using the errors of the data given by Schmucker (1970) and assuming an estimation error of 20%, Fig. 4 shows the resolution matrix for the model of Fig. 3. It is seen that only high-conducting regions are clearly resolvable.

4. Conclusion

The inversion of two-dimensional structures is still at its beginning. Only when the possible anomalous conductivity structure is confined either to an undulated interface between an insulator and a perfect conductor or to a thin non-uniform sheet, exact methods exist.

The more realistic case of a thick anomalous domain can in the moment only be attacked by linearization using either the Backus-Gilbert procedure or the closely related generalized matrix inversion. In a very preliminary investigation of two-dimensional inversion by linearization the latter method has been applied, because it is particularly suited for discrete variables. The disadvantage of this method compared with the Backus-Gilbert procedure is that the resolution function is not so clearly normalized to allow an easy interpretation. Much work is still necessary, in particular the non-linear effects are not yet clearly investigated.

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