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ANALYTICAL SOLUTIONS TO GLOBAL AND LOCAL PROBLEMS OF ELECTROMAGNETIC INDUCTION IN THE EARTH

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This review of analytic solutions is divided into two parts. The first part reviews electromagnetic induction in radially symmetric distributions of conductivity, and is appropriate to the study of global problems. In the second part, local problems of a specific nature are considered, the model being a half-space conductor with at least one lateral discontinuity separating regions of different uniform conductivities. In some problems, an approximate surface-boundary condition is used, and it is shown that the accuracy of the solutions has yet to be determined satisfactorily.

1. Introduction

Is it worth determining analytic solutions to electromagnetic induction problems? The model conductivities that can be considered are exceedingly simple and bear little relation to the real earth. The mathematical difficulties encountered are often great, and when solutions are determined they are usually expressed as special functions, as infinite series or in closed form for example as integral transforms. For analysing the solutions, or for comparison with measurements in a given situation, these solutions have to be evaluated approximately by numerical methods. Indeed, for some closed form solutions, even numerical evaluation is not possible. Why not be content, then, with solutions obtained by direct numerical integration of the relevant differential equations?

For the mathematician, part of the answer must surely lie in the elegance of analytic solutions. However the main reason for searching for such solutions is that, once found, they characterise the problem in terms of its parameters. The dependence of the solution on the various parameters can then be investigated, and special cases may arise when suitable limiting values are considered. When only numerical solutions are available for prescribed parameters, a great deal of computation is usually required to infer the dependence of the solution on any one parameter, and the investigation can never be exhaustive. For some applications, analytic solutions have to be evaluated numerically and this usually reduces to the evaluation of special functions or the use of well-known techniques of numerical integration. In such cases quantitative estimates of the errors involved in the approximations can be given. Further, the boundary conditions at infinity play an important part in determining the form of analytic solutions, but in direct numerical integrations only a limited region of the problem can be considered and the boundary conditions will necessarily be approximate. It is important therefore to check numerical solutions against analytic solutions, where the latter are available for simple problems, before more complex situations are investigated. Finally, however, it should be remembered that these "exact" solutions relate to the mathematical problem under investigation, and the differential equations and boundary conditions will themselves only be an approximation to the physical situation.

This review has to be strictly limited in length and in material, and is supposed to complement the review papers presented at the first Workshop on Electromagnetic Induction held in Edinburgh in 1972. At that meeting, a review of induction in a plane-layered earth was presented by Weaver (1973), and a review of induction in thin sheets was given by Ashour (1973). The problems considered in these reviews have features associated with both global and local problems of electromagnetic induction. It seems reasonable, however, to divide this present review into two more or less distinct sections. In "global problems" I shall consider spherical models with radially symmetric conductivity distributions, and in local problems I shall concentrate on semi-infinite half-space models with lateral inhomogeneities in conductivity. The short preliminary section contains the basic electromagnetic induction equations in the quasi-static approximation. The notation is consistent in this review and may not be the same as in the original articles.

2. Basic equations

In the quasi-static induction approximation, Maxwell's equations take the form:

$$\text{curl}\, \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \tag{1}$$

and

$$\text{curl}\, \boldsymbol{H} = \boldsymbol{J} \tag{2}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field intensities respectively, $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{J}$ is the current density and t is time. The constitutive equations are:

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{3}$$

and

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{4}$$

where σ and μ are the conductivity and permeability respectively.

Displacement currents have been neglected in eq. 2 under the assumption that, for most geomagnetic applications, the rate of variation of the fields is sufficiently small. Eq. 1 implies that:

$$\text{div}\, \boldsymbol{B} = 0 \tag{5}$$

and eq. 2 that:

$$\text{div}\, \boldsymbol{J} = 0 \tag{6}$$

Eq. 6 is true only within the approximation that has neglected displacement currents.

Using eqs. 1–4, $\boldsymbol{E}$ and $\boldsymbol{H}$ are found to satisfy:

$$\text{curl curl}\, \boldsymbol{E} = -\mu\sigma\dot{\boldsymbol{E}} + 1/\mu \text{ grad } \mu \wedge \text{curl}\, \boldsymbol{E} \tag{7}$$

and

$$\text{curl curl}\, \boldsymbol{H} = -\mu\sigma\dot{\boldsymbol{H}} + 1/\sigma \text{ grad } \sigma \wedge \text{curl}\, \boldsymbol{H} \tag{8}$$

where dots denote differentiation with respect to time.

The conductivity σ varies throughout the earth, and makes eq. 8 difficult to deal with. However, there are reasons for assuming that μ is constant, and with this assumption, eq. 7 takes the more amenable form:

$$\text{curl curl}\, \boldsymbol{E} = -\mu\sigma\dot{\boldsymbol{E}} \tag{9}$$

and the magnetic field can be obtained via eq. 1. Eq. 9 is therefore to be solved in the conducting part of the earth.

In any non-conducting region, eqs. 2 and 4 imply:

$$\boldsymbol{H} = -\text{grad}\, \Omega \tag{10}$$

where Ω is a scalar point function (the magnetic scalar potential) and then eqs. 3 and 5 imply:

$$\nabla^2 \Omega = 0 \tag{11}$$

Across any boundary marking a discontinuity in σ, the field equations 1 and 2 imply the continuity of (1) the tangential component of $\boldsymbol{E}$; (2) the tangential component of $\boldsymbol{H}$; and (3) the normal component of $\boldsymbol{B}$.

The above statement of the electromagnetic induction problem follows Price (1967). An alternative form can be obtained by noting that eq. 5 implies the existence of a magnetic vector potential $\boldsymbol{A}$, where:

$$\boldsymbol{B} = \text{curl}\, \boldsymbol{A} \tag{12}$$

This form of the analysis is described by Lahiri and Price (1939) and Chapman and Bartels (1940).

3. Global problems

In considering properties of the earth as a whole, it is convenient to take a spherical polar coordinate system (r, θ, φ) with origin at the earth's centre. In the non-conducting region outside the earth, the magnetic scalar potential Ω satisfies Laplace's equation 11 and the appropriate solution is:

$$\Omega = a \sum_{n=0}^{\infty} \left[e_n(t) \left(\frac{r}{a}\right)^n + i_n(t) \left(\frac{r}{a}\right)^{-n-1} \right] S_n \tag{13}$$

where a is the radius of the earth and S_n is a surface harmonic of degree n satisfying:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2}+n(n+1)S_n=0 \quad (14)$$

The terms involving $e_n(t)$ represent the field of external origin so that the coefficients may be regarded as known functions of time. The terms involving $i_n(t)$ represent the field arising from within the earth, and the coefficients have to be determined for a given model σ. The internal part has been interpreted as arising from induction by the external part since the paper of Schuster (1889). However, the causal nature of this relationship seems not to have been exploited until very recently (Bailey, 1970).

On account of the linearity of Maxwell's equations, the terms comprising the inducing field may be considered separately, the equations solved for the corresponding induced part, and then the total solution determined by superposition. Accordingly, the solution of eq. 9 corresponding to a typical harmonic in the inducing field is required for various model conductivities $\sigma(r, \theta, \varphi)$. The permeability μ also appears in eq. 9 but this is taken to be constant throughout the earth, and usually equal to the value in free space, μ_0.

The conductivity $\sigma(r, \theta, \varphi)$, in these global problems, represents some smoothed function of the coordinates and it is argued that in the real earth σ is likely to vary more with r than with either of θ or φ. The approximation $\sigma = \sigma(r)$ then makes the equations manageable. Two cases of induction arise, one when the temporal part of the inducing field is periodic, and the other when it is aperiodic. Table I gives a summary of the spherically symmetric models that have been investigated, together with the inducing fields that have been considered. The method of solution is similar for each case in Table I, and it suffices to indicate the form of the analysis with reference to the work of Lahiri and Price (1939) and Price (1967). The other examples cited are then special cases.

Consider a spherical conductor radius qa conductivity $\sigma(r)$, surrounded by a dielectric shell of thickness $(1-q)a$. Let the normalised radius in the conductor be $\rho = r/qa$. Then outside the conductor, the magnetic scalar potential Ω may, from eq. 13, be expressed as the sum of spherical harmonics of the form:

TABLE I

Investigations of electromagnetic induction in spherically symmetric distributions of conductivity

Investigator	Type	Model			Inducing field	Model parameters
		radius	conductivity	permeability		
Lamb, 1883	sphere	qa	k	μ_0	periodic	k, q
Chapman and Whitehead, 1923	sphere (i)	qa	k	μ	periodic	k, q, μ
	sphere + thin shell (ii)	qa a	k K	μ_0 μ	periodic	k, q, μ, K
Price, 1930	sphere	qa	k	μ_0	aperiodic, axially symmetric	k, q
Price, 1931	sphere	qa	k	μ	aperiodic	k, q, μ
Chapman and Bartels, 1940 (useful review)	sphere	qa	k	μ	periodic and aperiodic	k, q, μ
Lahiri and Price 1939	sphere (i)	qa	$k(r/qa)^{-m}$	μ formulae for μ_0	periodic and aperiodic	k, q, m
	sphere + thin shell (ii)	qa a	$k(r/qa)^{-m}$ K	μ_0 μ_0	periodic and aperiodic	k, q, m, K

K, k, q, a, μ, μ_0 and m are constants.

$$\Omega_n = a[e_n(t)(q\rho)^n + i_n(t)(q\rho)^{-n-1}]S_n \tag{15}$$

In the conducting region, the electric field may be expressed in terms of toroidal and poloidal vector fields, and these are of the form curl Ψr and curl curl Φr respectively (Morse and Feshbach, 1953). Ψ and Φ are scalar functions of (r, θ, φ). The poloidal electric field has no magnetic field outside the conductor and cannot be excited by any external inducing field. It does not therefore enter into the induction problem. Substituting the toroidal solution into eq. 9, the equation for Ψ is obtained as:

$$\nabla^2\Psi - \mu\sigma(r)\dot{\Psi} = 0 \tag{16}$$

Corresponding to the surface harmonics S_n in eq. 15 a separation of variables in eq. 16 shows that there are solutions of the form:

$$\Psi_n = aR_n(t, \rho)S_n(\theta, \varphi) \tag{17}$$

where R_n satisfies:

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial R_n}{\partial\rho}\right) = [n(n+1) + \mu a^2\rho^2\sigma(\rho)\partial/\partial t]R_n \tag{18}$$

Thus the electric field in the conductor is:

$$\begin{aligned} E &= \text{curl } [aR_n(t, \rho)S_n(\theta, \varphi)r] \\ &= aR_n(t, \rho)r \wedge \text{grad } S_n \end{aligned} \tag{19}$$

The solution in the dielectric and in the conductor has now to be matched at the boundary $\rho = 1$, and the continuity conditions are satisfied if:

$$R_n(t, 1) = \mu_0\left[-\frac{q^n}{n+1}\frac{de_n}{dt} + \frac{q^{-n-1}}{n}\frac{di_n}{dt}\right] \tag{20}$$

and

$$\left[\frac{\partial}{\partial\rho}\{\rho R_n(t, \rho)\}\right]_{\rho=1} = -\mu\left[q^n\frac{de_n}{dt} + q^{-n-1}\frac{di_n}{dt}\right] \tag{21}$$

Eq. 20 makes both the tangential component of $\boldsymbol{E}$, and the normal component of $\boldsymbol{B}$ continuous and eq. 21 ensures the continuity of the tangential component of $\boldsymbol{H}$.

The problem is now reduced to the determination of $R_n(t, \rho)$ and hence $i_n(t)$. There are in general two linearly independent solutions of eq. 18, so that R_n will be arbitrary to within two constants. However, only one solution will be analytic at the origin, so that if the conductor occupies the region $0 \leqslant \rho \leqslant 1$, the two boundary conditions 20 and 21 are sufficient to determine uniquely the remaining constant from eq. 18 together with i_n. (Any further spherical discontinuity between 0 and 1 results in two more constants and another two boundary conditions.)

What remains is to find solutions to eq. 18 for prescribed conductivity distributions $\sigma(\rho)$, and it is because of the complexity of this equation that only very simple distributions of σ can be dealt with analytically.

The simplest approach seems to be to replace $\partial/\partial t$ by a parameter p, and solve eq. 18 as an ordinary differential equation for $R_n(p, \rho)$. Periodic inducing fields may then be considered by writing $p = i\omega$ (where ω, the angular frequency, is real) and aperiodic fields by interpreting the solutions in terms of known results in the operational calculus (Lahiri and Price, 1939). An alternative method for aperiodic fields is afforded by determining the free current systems that can exist in the given sphere. The boundary conditions can then be satisfied by a particular solution of eq. 18 for the given inducing field, together with the appropriate sum of these free current systems. For a uniform sphere, this method is used in Lamb (1883), Price (1930) and Chapman and Bartels (1940).

The most general conductivity distribution so far considered analytically (Lahiri and Price, 1939) is:

$$\sigma(\rho) = k\rho^{-m} \tag{22}$$

where k and m are real constants and $k > 0$. With $m > 0$ distribution 22 represents an increase of conductivity with depth. The simpler case of a uniformly conducting sphere is obtained by taking $m = 0$.

With the distribution 22, eq. 18 becomes:

$$\frac{d}{d\rho}\left(\rho^2\frac{dR_n}{d\rho}\right) = [n(n+1) + \zeta^2\rho^{-m+2}]\,R_n \tag{23}$$

where

$$\zeta^2 = \mu akp \tag{24}$$

For $m \neq 2$, the substitutions:

$$R_n = \rho^{-\frac{1}{2}}W_n\ , \qquad z = \frac{2\zeta}{|m-2|}\rho^{1-m/2}\ , \qquad \nu = \frac{2n+1}{|m-2|} \tag{25}$$

reduce eq. 18 to the Bessel equation of order ν whilst for $m = 2$, eq. 18 is homogeneous linear. Accordingly, in the three cases $m < 2$, $m = 2$ and $m > 2$, the solutions of eq. 18 for R_n regular at $\rho = 0$ are:

m	$m < 2$	$m = 2$	$m > 2$	
solution	$\rho^{-\frac{1}{2}} I_\nu(z)$	$\rho^{-\frac{1}{2}+s}$	$\rho^{-\frac{1}{2}} K_\nu(z)$	(26)

where I_ν and K_ν are the modified (or hyperbolic) Bessel functions of the 1st and 2nd kinds, respectively (Morse and Feshbach, 1953), of order ν and $s = [n(n+1) + \frac{1}{4} + \zeta^2]^{\frac{1}{2}}$. In the case of a uniformly conducting thick shell, with $m = 0$ in distribution 22, it is convenient to write $\xi^2 = -\zeta^2$, and then the general solution of eq. 18 is:

$$R_n = A j_n(\xi\rho) + B n_n(\xi\rho) \qquad (27)$$

where j_n and n_n are spherical Bessel functions of the 1st and 2nd kinds, respectively (Morse and Feshbach, 1953). A and B are arbitrary constants and for a complete sphere (including the origin) $B = 0$, leading to the same regular solution as above.

This completes the formal solution, and the electric and magnetic fields together with the induced current distribution can all be obtained from the above equations. The interpretation of these solutions for periodic and aperiodic inducing fields and many asymptotic formulae useful for computation are given by Lahiri and Price (1939). The case $m = 0$ is well documented by Chapman and Bartels (1940).

The above type of model can be made more elaborate by the inclusion of a uniformly conducting thin surface shell (radius a) surrounding the inner conductor and dielectric and the extension is straightforward (see Table I).

Analytic solutions have so far only been obtained when the conductivity is of the special form 22. A method of solving eq. 18 for a more general conductivity distribution is to divide the sphere into concentric thick shells each of different conductivity, the conductivity within any one shell being constant. This has the advantage that within each shell the analytic solution 27 is known apart from, in general, two constants, and these are determined by the conditions at the boundaries of the shell. It is possible to solve these boundary equations to yield the solution in each layer, but even for two or three layers the solutions are cumbersome. The equations can be solved numerically however and Banks (1969) gives an elegant matrix method of solution.

The solutions that I have so far described mainly find their applications in model fitting. That is, some quantity is determined by measurement at the surface of the earth, and this same quantity is calculated from the solution for given distributions $\sigma(\rho)$. When the observed and calculated values agree, it is possible that the model $\sigma(\rho)$ may resemble some of the features of the real earth. One such response measure frequently used in global studies is the ratio of internal to external parts (i_n/e_n) in a given spherical harmonic in the expression for the inducing field 13. Taking the case of a periodic inducing field, this response measure may be expressed in terms of the amplitude ratio and phase difference of these internal and external parts.

Instead of deriving these quantities from the solution for a given $\sigma(\rho)$, Eckhardt (1963) adopted a new approach and determined the differential equation that this ratio i_n/e_n ($= V_n$) satisfies. For a periodic inducing field of angular frequency ω eqs. 20 and 21 applied at the surface of any subsphere of radius ρ enable $V_n(\rho)$ to be written:

$$V_n = \frac{n}{n+1}\left[\frac{\rho(\mathrm{d}R_n/\mathrm{d}\rho) - nR_n}{\rho(\mathrm{d}R_n/\mathrm{d}\rho) + (n+1)R_n}\right] \qquad (28)$$

Now make the substitution $Z_n = (1/R_n)(\mathrm{d}R_n/\mathrm{d}\rho)$ so that:

$$Z_n = \frac{1}{\rho}\left[\frac{n^2 + (n+1)^2 V_n}{n - (n+1)V_n}\right] \qquad (29)$$

This substitution also reduces eq. 18 to Riccati's equation, whence V_n satisfies:

$$\frac{\mathrm{d}V_n}{\mathrm{d}\rho} = -\frac{k^2\rho(n+1)}{n(2n+1)}\left[V_n - n/(n+1)\right]^2 - \frac{2n+1}{\rho}V_n \qquad (30)$$

where $k^2(\rho) = -\mu a^2 \mathrm{i}\omega\sigma(\rho)$.

Eckhardt then considers the solution characteristics for V_n from eq. 30, and in one case, determines an analytic solution for V_n which yields a conductivity distribution. This conductivity distribution is of the form $\sigma = K/r^2$, where K is constant and does not form a new solution, having been previously considered by Lahiri and Price (1939). The interesting equation for V_n, however, forms the basis of the now celebrated paper by Bailey (1970). Writing the linear dependence of i_n on e_n in the most general form:

$$i_n = \int_{-\infty}^{\infty} K_n(\tau) e_n(t-\tau)\mathrm{d}\tau \qquad (31)$$

where $K_n(\tau)$ is the impulse response of the earth to a given harmonic mode, Bailey exploits the fact that $K_n(\tau)$ must be causal at all radii (that is $K_n(\tau) = 0$ for $\tau < 0$) to determine a non-linear partial integro-differential equation for Φ_n which is related to V_n by:

$$\Phi_n = \rho^{2n+1} [V_n - n/n + 1] \tag{32}$$

Bailey then proves that under certain conditions a knowledge of Φ_n for all ω at some ρ (e.g. the surface of the earth) determines the conductivity uniquely as a function of ρ. Although it is not possible to obtain such a knowledge of Φ_n, the work does show that in principle it is possible to determine the radial conductivity distribution from surface measurements. A method of performing the inversion procedure is given.

An alternative form of the inversion procedure is formulated by Jady (1974). Instead of determining conductivities between fixed concentric boundaries, Jady uses a variational technique to determine the boundary and conductivity (ρ_0, σ_0) in a model consisting of a perfectly conducting inner sphere radius ρ_0, surrounded by a thick shell of uniform conductivity σ_0. The analytic solution for such a model with a fixed boundary is easily determined, but the virtue of Jady's approach is to determine the boundary as part of the solution (an inverse problem) rather than solve the direct problem for a number of different boundary positions (a model-fitting problem).

A response measure much used in local problems of electromagnetic induction is the magnetotelluric relation E_θ/H_φ (or $-E_\varphi/H_\theta$). Srivastava (1966) examines this relation in the case of a spherical earth for: (i) the Lahiri-Price conductivity distribution; and for (ii) a model divided into thick concentric spherical shells each with a different conductivity. For (i) the solutions 26 provide the magnetotelluric relation explicitly, while for (ii) eq. 27 and the boundary conditions at each interface determine a recurrence relation. Srivastava shows that for realistic conductivities and periods of magnetic variations up to one day, the curveture of the earth may be neglected and the magnetotelluric relations are as derived for a plane-layered earth.

Schmucker (1970) has also considered induction in a sphere composed of concentric shells of different uniform conductivities and by using relations between spherical Bessel functions (eq. 27) and their derivatives (required for expressions of the magnetic field) has derived a much simplified form of Srivastava's recurrence relation for a function $G(r)$ which involves the ratio of certain of the spherical Bessel functions. The application of this recurrence formula provides the surface value $G(a)$ and simple formulae then give the ratios of internal to external parts for the tangential and radial field components, as well as the magneto-telluric relations. Schmucker also gives a concise formulation for the attenuation of the magnetic field with depth, and also some approximations to the function $G(r)$ which are useful in numerical calculations.

4. Local problems

The discussion in this section will be confined to "half-space" models of the earth, in which the conductivity changes laterally. In a cartesian coordinate system (x, y, z), with the z-axis vertically down, the conductor occupies the region $z > 0$, and the inhomogeneity is confined to a limited portion (or plane) of the conductor. This gives the problem its local character. However the total magnetic field is often of global dimensions, and hence over the local region of interest may sometimes be considered uniform. Care must then be exercised in the formulation of such a problem (Price, 1950, 1964).

Much of the literature is concerned with modelling lateral inhomogeneities in conductivity by appropriate two-dimensional conductors. In principle, two approaches can be made. In the first, boundaries within the conductor separate regions of different, but uniform, conductivities. Maxwell's equations must be solved in each region, and the appropriate continuity conditions have to be satisfied at each boundary. In the second, the conductivity is specified as a continuously variable function of the spatial coordinates. I shall concentrate on the former approach.

For two-dimensional problems, let all quantities be independent of the x coordinate, and let all field quantities vary in time as $\exp(i\omega t)$. Then, with $\partial/\partial x \equiv 0$ and $\mu = \mu_0$, Maxwell's equations 1 and 2 in component form become:

$$\partial E_z/\partial y - \partial E_y/\partial z = -i\omega\mu_0 H_x \tag{33}$$

$$\partial E_x/\partial z = -i\omega\mu_0 H_y \tag{34}$$

$$\partial E_x/\partial y = i\omega\mu_0 H_z \tag{35}$$

$$\partial H_z/\partial y - \partial H_y/\partial z = \sigma E_x \tag{36}$$

$$\partial H_x/\partial z = \sigma E_y \tag{37}$$

$$-\partial H_x/\partial y = \sigma E_z \tag{38}$$

where $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{H} = (H_x, H_y, H_z)$.

The equations conveniently separate into two distinct cases, since eqs. 34,35 and 36 involve only E_x, H_y and H_z whereas eqs. 33, 37 and 38 involve only E_y, E_z and H_x:

(1) The first case is that of $\boldsymbol{E}$ polarisation (the electric field being parallel to any discontinuity in σ) and may be specified by $\boldsymbol{E} = (E, 0, 0)$ and $\boldsymbol{H} = (0, H_y, H_z)$. The solution may be determined through the scalar E that satisfies Helmholtz equation:

$$\nabla^2 E = i\eta^2 E \tag{39}$$

subject to the appropriate boundary conditions, where $\eta^2 = \mu_0\sigma\omega$ and ∇^2 is the two-dimensional Laplacian operator in y and z. $\boldsymbol{H}$ may then be obtained from eqs. 34 and 35.

(2) The second case, $\boldsymbol{H}$ polarisation, is specified by $\boldsymbol{H} = (H, 0, 0)$ and $\boldsymbol{E} = (0, E_y, E_z)$. The scalar H satisfying:

$$\nabla^2 H = i\eta^2 H \tag{40}$$

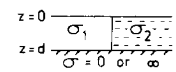
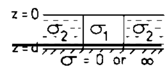
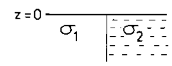
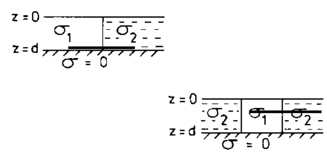
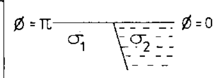
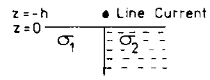
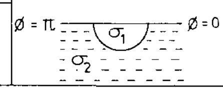
now determines the problem and $\boldsymbol{E}$ is obtained from eqs. 37 and 38.

The two cases are not as similar as they might at first appear, an important difference arising from the boundary condition on the surface $z = 0$. For $\boldsymbol{H}$ polarisation eqs. 37 and 38 show that in the non-conducting region $\boldsymbol{H}$ is independent of y and z. Hence $\boldsymbol{H}$ is uniform throughout $z = 0$. As $\boldsymbol{H}$ is continuous across the surface, the boundary condition on eq. 40 is $\boldsymbol{H}$ = constant on $z = 0$, irrespective of the conductivity in $z > 0$. For $\boldsymbol{E}$ polarisation, no such simplification occurs.

The earliest study seems to be that of D'Erceville and Kunetz (1962) who considered the $\boldsymbol{H}$ polarisation case for two models. (See Table II for diagrams of the models considered in this section.) In both models a horizontal layer $0 \leqslant z \leqslant d$, composed of two media of different uniform conductivities σ_1 and σ_2 in contact along the vertical plane $y = 0$, overlies a semi-infinite half-space $z > d$. In one model the half-space is perfectly conducting, in the other, non-

TABLE II

Investigations of electromagnetic induction in non-uniform half-space conductors (two-dimensional problems)

Investigator	Model	Polarisation
D'Erceville and Kunetz (1962)		H
Rankin (1962)		H
Weaver (1963)		H, E*
Treumann (1970 b)		E** E**
Hvozdara (1968) Geyer (1970)		H, E*
Weaver and Thomson (1972)		E**
Hvozdara (1969)		H, E*

For $\boldsymbol{E}$ polarisation, * denotes a zero'th-order approximation, ** denotes a first-order approximation.

conducting and as $d \to \infty$ both models tend to the same configuration. The boundary conditions at the vertical interface $y = 0$ are that $\boldsymbol{H}$ and the tangential component of $\boldsymbol{E}$ are continuous.

As noted above, in the non-conductor, for $\boldsymbol{H}$ polarisation, the magnetic field is uniform. In each conducting region in the layer, the magnetic field is assumed to be composed of an undisturbed field calculated as if that region were of infinite horizontal extent, together with a disturbance field arising from the discontinuity at $y = 0$. The undisturbed field in each region satisfies a one-dimensional diffusion equation (from eq. 40) and hence is known to within two constants. These two constants are determined by the boundary conditions on $z = 0$ and $z = d$. D'Erceville and Kunetz then expand the disturbance fields as Fourier sine series in z, the coefficients appropriate to each region being determined from the two boundary conditions on $y = 0$. Explicit formulae are given for the Fourier coefficients for the two cases under consideration, and also for the magnetotelluric relation E_y/H for the limit $d \to \infty$.

The model was extended by Rankin (1962) to enable the effect of a dike to be considered. In the layer of thickness d in his model, the region $-l/2 \leqslant y \leqslant l/2$ has conductivity σ_1, the infinitely extending regions on either side have conductivity σ_2. The same method as above is employed, the general solution for H in each region being written as the solution for a plane layer together with a disturbance term, the appropriate quantities again being expressed as Fourier sine series. For each sine term, the disturbance fields require one coefficient in $y < -l/2$, one in $y > l/2$ and 2 in $-l/2 < y < l/2$. These four coefficient are determined by the two boundary conditions on the two vertical discontinuities at $y = \pm l/2$.

The first attempt at discussing both $\boldsymbol{H}$ and $\boldsymbol{E}$ polarisation for a single model was made by Weaver (1963). His model was the limiting form of those of D'Erceville and Kunetz and so consisted of two quarter spaces in contact along the vertical plane $y = 0$. As noted by D'Erceville and Kunetz, for $\boldsymbol{H}$ polarisation the limiting case can be treated by replacing the Fourier sine series by Fourier sine transforms. Weaver defines the Fourier sine transform of a function $\Phi(z)$ by:

$$\tilde{\Phi}(\xi) = \sqrt{2/\pi} \int_0^\infty \Phi(z) \sin \xi z dz \quad (41)$$

and transforms eq. 40, with the use of the boundary conditions on $z = 0$ and as $z \to \infty$, to:

$$\partial^2 \tilde{H}/\partial y^2 = (\xi^2 + i\eta^2)\tilde{H} - A\xi\sqrt{2/\pi} \quad (42)$$

where A is the constant value of H on $z = 0$. Integration of eq. 42 with respect to y yields $\tilde{H}_1$ and $\tilde{H}_2$ in each region, apart from some function, say $f(\xi)$. H_1 and H_2 are obtained by applying the inverse transform to the solution, and finally the unknown function $f(\xi)$ is determined from the boundary condition that the tangential component of $\boldsymbol{E}$ is continuous. The solutions for H_1 and H_2 are expressed in terms of Fourier sine integrals and as required are the limiting form of the solutions of D'Erceville and Kunetz as $d \to \infty$.

Weaver then discusses the $\boldsymbol{E}$ polarisation case, first making some assumptions about conditions at the boundary $z = 0$. He considered the reflection of plane waves normally incident on a uniformly conducting half-space, and concluded that the horizontal component of the magnetic field at the surface is independent of conductivity and frequency. Using eq. 34, Weaver's boundary condition is:

$$\partial E_1/\partial z = \partial E_2/\partial z = \text{constant} \qquad \text{on } z = 0 \quad (43)$$

Also continuity of the magnetic field and tangential electric field on $y = 0$ requires:

$$E_1 = E_2 \qquad \text{on } y = 0 \quad (44)$$

$$\partial E_1/\partial y = \partial E_2/\partial y \qquad \text{on } y = 0 \quad (45)$$

The above relations are sufficient to determine the electric and magnetic fields within the conductor and these may be obtained by using the same method as for $\boldsymbol{H}$ polarisation, except that Fourier cosine transforms are used in place of Fourier sine transforms. However the field in $z < 0$ was not analysed, and more recent investigations (e.g., Jones and Price, 1970) have shown that condition 43, implying that H_y is constant on $z = 0$, is not appropriate. It transpires that Weaver's solution for $\boldsymbol{E}$ and $\boldsymbol{H}$ in the conductor can be used to generate an approximate solution in $z < 0$, and hence an approximation to the fields on the surface. In its present form however, the solution is not satisfactory. This situation, that of taking H_y constant on $z = 0$ for $\boldsymbol{E}$ polarisation, occurs elsewhere in the literature so that for these "solutions", H_y and the magnetotelluric relation E/H_y are not satisfactorily determined. Such analyses are marked with a single asterisk in Table II.

By considering the problem of plane waves incident on non-uniform conductors, Mann (1970) has derived a successive approximation scheme whereby the solutions, in the conducting and non-conducting regions, are expressed as power series in a certain parameter δ. This parameter is the ratio of the skin depth of the conductor to the wavelength of the inducing field. The method is proposed as a simpler and more accurate alternative to the Leontovich boundary conditions for radiation fields, which only determine the solution to $O(\delta^2)$. Using this new method, provided the equations can be solved at each stage, the solution can be determined to any accuracy in δ. In Mann's method, a starting approximation is obtained by considering the half-space to be perfectly conducting. For a given incident wave, the solution for the electric and magnetic fields in $z < 0$, $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$, may then be obtained analytically. The affix n denotes the nth approximation. In the region $z > 0$, where the resistivity is zero, the magnetic and electric fields $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ vanish. This completes the zero'th approximation, and from it the magnetic field as $z \to 0-$ may be determined. The model conductivity (finite) is now assumed, and consequently the tangential component of the magnetic field is continuous across the boundary $z = 0$. Using this tangential component as the boundary condition on $z = 0+$, the first correction terms $\boldsymbol{E}^1$ and $\boldsymbol{H}^1$ may be determined in the conductor. At the surface $z = 0$, the first correction term to the electric field is continuous, and so provides the boundary value enabling $\boldsymbol{E}^1$ to be determined in the non-conductor. From $\boldsymbol{E}^1$, the corresponding magnetic field $\boldsymbol{H}^1$ can be found and now the first approximation to the electric field $\boldsymbol{E}^0 + \boldsymbol{E}^1$, and to the magnetic field, $\boldsymbol{H}^0 + \boldsymbol{H}^1$, is known everywhere. The next approximation commences with the boundary value provided by $\boldsymbol{H}^1$ and so the successive approximation scheme is defined. The method is convergent provided the far field of the term of $0(1)$ in the non-conducting region is not affected by the finite conductivity.

As an example, Mann (1970) considers the case of a plane wave ($\boldsymbol{E}$ polarised) incident on a half-space composed of two quarter spaces of different uniform conductivities in contact in a vertical plane. The zero'th approximation generates the boundary condition that the tangential component of the magnetic field is constant along the boundary $z = 0$. In this form the problem is that considered by Weaver (1963), and hence $\boldsymbol{E}^1$ and $\boldsymbol{H}^1$ are determined for $z > 0$. Using the boundary value of $\boldsymbol{E}^1$, Mann gives the solution for $\boldsymbol{E}$ in $z < 0$ in integral form. In this context, Weaver's solution is seen as the first approximation within the conductor, but the zero'th approximation in $z < 0$.

Weaver and Thomson (1972) show that Mann's perturbation method for incident electromagnetic waves can be used to solve problems of electromagnetic induction and they interpret the perturbation parameter as the ratio of the skin depth to a "characteristic length" representative of the region under consideration. Their model is again the half-space divided by a vertical discontinuity at $y = 0$ into two regions of different uniform conductivities. As an inducing field they consider that due to a periodic line current, of magnitude $I \exp(i\omega t)$ at $y = 0$, $z = -h$ ($\boldsymbol{E}$ polarisation). In the zero'th approximation, the conductivity is assumed infinite, hence the induced field in $z < 0$ may be determined simply by image methods. The sum of these induced terms with the original source terms provides the zero'th approximation $\boldsymbol{E}^0$, $\boldsymbol{H}^0$ in $z < 0$. The horizontal component of the magnetic field is found to be:

$$H_y^0 = \frac{4Ih(z^2 - y^2 - h^2)}{[y^2 + (z - h)^2]\,[y^2 + (z + h)^2]} \tag{46}$$

Inside each quarter space, now taken to have finite conductivity, the electric field $\boldsymbol{E}$ satisfies Helmholtz equation with the boundary condition, from eqs. 34 and 46:

$$\left(\frac{\partial E}{\partial z}\right)_{z=0} = \frac{4i\omega Ih}{y^2 + h^2} \tag{47}$$

Using Fourier cosine transforms the solution for E is obtained, the result being somewhat complicated. The value of this expression on the boundary, $E^1(y, 0)$, being continuous at $z = 0$, provides the boundary condition for E^1 in the region $z < 0$. Since in this region E^1 satisfies Laplace's equation, it is given by Poisson's integral for the half plane:

$$E^1(y, z) = -\frac{z}{\pi} \int_{-\infty}^{\infty} \frac{E^1(\xi, 0)}{(\xi - y)^2 + z^2}\, d\xi \tag{48}$$

Weaver and Thomson manage to determine $E^1(y,z)$ from eq. 48 and this, together with its derivative with respect to z gives the first approximation to the electric and magnetic fields everywhere. It seems most unlikely that the equations could be solved analytically to yield higher-order terms.

Numerical calculations are made of the electric field and the vertical and horizontal components of the magnetic field at $z = 0$ and these are plotted for four conductivity contrasts σ_2/σ_1 representative of most problems of geophysical interest. Over the discontinuity the variation in $\boldsymbol{H}$ and $\boldsymbol{E}$ is quite marked and increases with increasing conductivity contrast. The accuracy of the solutions obtained using Mann's scheme may be examined to some extent by taking $\sigma_1 = \sigma_2$ and comparing $E^1(y, 0)$ from eq. 48 to the exact solution for a uniform half-space. The error in eq. 48 is then found to be $0(\delta/h)$ where δ is the skin depth. When $h = 10\delta$, numerical calculations indicate that eq. 48 is correct to within 10% at $y = 2\delta$, the accuracy improving as y increases.

The case of a uniform inducing field is treated by taking the limiting form of the solution as $h \to \infty$, and E, H_y and H_z are again plotted on $z = 0$ for the same conductivity contrasts. For one of these contrasts, $\sigma_2 = 10\sigma_1$, calculations are also made of the electric and magnetic fields at various heights above the surface $z = 0$. In this way, the solution is directly compared to the work of Jones and Price (1970) who are developing numerical methods for dealing with this, and other more complicated models, and who use the boundary condition that the magnetic field is uniform at a certain height above the surface. The results for the magnetic field are not compatible, however, and Weaver and Thomson suggest that the Jones-Price boundary condition has not been applied sufficiently far from the discontinuity. They suggest a more suitable height at which the perturbation field should be negligible. Jones (1972) has reformulated the numerical method, and has calculated the perturbation field due to the discontinuity at various heights above the surface. He finds the perturbation field to be small at the increased height suggested by Weaver and Thomson, in accordance with their values there, but it is not immediately obvious whether the solution at the surface $z = 0$ has remained the same as in Jones and Price (1970) or is in closer agreement with Weaver and Thomson.

In this case of a uniform inducing field, it is not entirely clear how the accuracy of the analytic solution is to be inferred, since the "characteristic length" can no longer be taken as the (infinite) height of the line current. Perhaps a characteristic length is the horizontal extent over which the anomalous field due to the discontinuity falls to $1/e$ of its value at the discontinuity. The results of Jones and Price and of Weaver and Thomson may well agree to this sort of accuracy.

Extending the range of models, Hvozdara (1968) and Geyer (1972) considered $\boldsymbol{H}$ and $\boldsymbol{E}$ polarisation for a half-space where two different conductivities were separated by a sloping plane interface. In Hvozdara's model, one of the conductivities is infinite. Employing cylindrical coordinates (r, ϕ, z) with the z-axis parallel to the plane of the interface, the surface of the conductor is defined by $\phi = 0$ and $\phi = \pi$, and the interface by $\phi = \alpha$ for some α. For $\boldsymbol{H}$ polarisation, Helmholtz equation 40, written in cylindrical coordinates, may be solved using a Green's function for each wedge space. However, the integrands cannot be matched at $\phi = \alpha$ to directly yield the required coefficients as the radial dependence is in the form of a Bessel function with different arguments on either side of the discontinuity. Instead the boundary condition on $\phi = \alpha$, together with the surface condition of a constant tangential magnetic field, results in a set of four simultaneous singular integral equations. The Lebedev-Kontorovich transform enables the problem to be solved analytically, but the expressions do not readily lend themselves to numerical evaluation. The problem is simpler for an infinite conductivity contrast, and Geyer gives the solution for the case $\sigma_2 \to \infty$, and also numerical methods of evaluating the resulting integrals. Hvozdara's results are similar, but only the magnitude of the solutions is evaluated.

Using the same method, Hvozdara (1968) and Geyer (1972) investigate the $\boldsymbol{E}$ polarisation case, and again use the condition that the tangential magnetic field is constant along the surface $\phi = 0$, $\phi = \pi$. The solution for the electric field and vertical component of the magnetic field, in the light of Mann's paper, are seen to be the first approximation, but the horizontal magnetic field at the surface remains as the (constant) zero'th approximation. The solutions are again evaluated numerically for an infinite conductivity contrast.

An alternative representation of the solution of the equations governing electromagnetic induction in thick sheets and half-spaces is given by Treumann (1970a). He concentrates on the $\boldsymbol{E}$ polarisation case and in cartesian coordinates his basic, two-dimensional, model consists of a sheet of thickness d in which regions of different uniform conductivities are separated by vertical boundaries y = constant. In $z > d$ the conductivity is zero. Within each region of uniform conductivity E satisfies Helmholtz equation and so Green's theorem can be used to relate E (anywhere) in the conductor in terms of the normal derivative of E on all the boundary surfaces. Thus:

$$E = \iint_{S_0} \frac{\partial E}{\partial n_0} \Gamma_k \, \mathrm{d}S_0 \tag{49}$$

where n_0 denotes the coordinate normal to the surface S_0, and Γ_k is Green's function of the second kind satisfying the inhomogeneous Helmholtz equation. Treumann expands Γ_k as a series of eigenfunctions, noting that in the case of vertical boundaries a finite number of coefficients have to be determined for each eigenfunction, whereas for an inclined boundary an infinite system of equations has to be solved. Solutions for Γ_k are given for the case of 1, and for 2, vertical discontinuities in the thick layer.

Outside the conductor, E satisfies Laplace's equation and thus contains parts of external (inducing) and internal (induced) origin. The problem is solved once the induced field is given as a function of the inducing field. However, Treumann shows that the boundary condition on $y = 0$ determines a singular integral equation, which on regularisation in the limit $d \to \infty$ and when there is only a discontinuity at $y = 0$, reduces to a quasi-linear Fredholm integral equation which has not yet been solved analytically.

For a thick plate (d finite), the regularisation would result in a system of quasi-linear Fredholm integral equations whose kernals cannot be given in closed form. It is clear that the $\boldsymbol{E}$ polarisation problem is not simple, and in a further paper, Treumann (1970b) constructs an approximate solution for the field components at the surface $z = 0$, using a method by Weidelt (1968). The procedure determines the same solution as Mann's (1970) perturbation scheme, although the solution for H_y is only specified on $z = 0$. Treumann determines the first approximations H_y^1 and H_z^1 at the surface for a uniform inducing field, and Weidelt has shown that in the limit $d \to \infty$, for a discontinuity at $y = 0$, the resulting expressions for these surface fields are identical to those of Weaver and Thomson (1972).

Numerical calculations are made for a uniform field for various sheet thicknesses d and for two conductivity contrasts σ_2/σ_1. Calculations are also made for a dike model (similar to that of Rankin (1962)). Only the surface values of H_y and H_z are presented and discussed, although E could have been obtained directly from eq. 49. This method extends the type of models that can be considered in the difficult $\boldsymbol{E}$ polarisation case.

The models reviewed in this section all have surface discontinuities in conductivity. One further problem of this nature is considered by Hvozdara (1969). In cylindrical coordinates (r, ϕ, z) the half-space has conductivity σ_1 in $0 \leqslant r \leqslant a$, and conductivity σ_2 in $r > a$, for some a. For $\boldsymbol{H}$ and $\boldsymbol{E}$ polarisation, the fields can be expressed as series of Bessel functions and the required coefficients determined by continuity conditions at $r = a$ and on the surface $\phi = 0, \pi$. Again for $\boldsymbol{E}$ polarisation, a constant magnetic field is assumed at the boundary.

Other two-dimensional problems of a local nature include the response of a conducting cylinder buried in material of zero conductivity, and possibly underlain by a semi-infinite conductor. For circular and elliptic cylinders a review is given by Rikitake (1966).

It seems to me that the major point requiring clarification in some of the above papers is the determination of the accuracy of solutions using Mann's approximation scheme. For the induction problem, where one is probably limited to obtaining the first approximation everywhere, only Weaver and Thomson seem to have discussed questions of accuracy, and then only for the case of induction by a line current. For induction by a uniform field, several workers have produced solutions "to first order", without specifying the accuracy, and these solutions are marked with two asterisks in Table II.

5. Conclusion

Dealing with global problems first, the determination of analytic solutions describing the induction of electric currents in radially symmetric distributions of conductivity of prescribed form seems to have ended with Lahiri and Price (1939). Now, numerical methods may be used to compute solutions for practically any conductivity distribution. It is generally concluded that the conductivity increases with depth in the earth's mantle, but the rate of increase is not well determined. It might be of some interest to solve the induction problem for a model where one of the parameters is the gradient of the conductivity with respect to depth. However, even in the simple model of a linear increase of conductivity with depth, the differential equations seem to be prohibitive.

For local problems, the situation is perhaps more open. No analytic solutions for $\boldsymbol{E}$ polarisation have yet been determined, and I have indicated that I think the accuracy of some approximate solutions should be determined quantitatively. An alternative approach to the problems involving discontinuous distributions of conductivity may be to consider the model as the limit of some continuously variable conductivity distribution. For example, if the induction equation could be solved for the distribution $\sigma = \frac{1}{2}\,[\sigma_1 + (\sigma_2 - \sigma_1) \arctan ky]$ where σ_1, σ_2 and k are constants, then the limiting model $k \to \infty$ would be that of two quarter spaces of conductivity σ_1 and σ_2. The equations seem to be too difficult for this distribution, but there may be other distributions for which the equations can be solved and which reduce to the same limit. Such solutions would be of great value in their own right, and would also provide clues as to the accuracy of the solutions for more complicated models.

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