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INDUCTION IN A LAYERED PLANE-EARTH BY UNIFORM AND NON-UNIFORM SOURCE FIELDS

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A review of electromagnetic induction in a multi-layered earth is built around a development of the general theory from first principles. Induction by transient and periodic fields and by dipole and electrojet sources are discussed and the method of complex images is briefly described. The review concludes with a discussion of induction by elementary harmonic sources whose non-uniformity is characterised by Price's ν-parameter and which include the uniform source field (ν=0) as a special case. The conditions under which the source may be assumed uniform for computing the surface impedance and other ratios of field components are examined.

1. Introduction

The induction problem to be reviewed can be summarised as follows. Let the earth be represented by an N-layered conducting half space occupying the region $z > 0$ in a right-handed Cartesian coordinate system (x,y,z), and let the n-th (n=1,2,...,N) layer $z_{n-1} < z < z_n$ ($z_0=0, z_N=\infty$) of thickness $d_n = z_n - z_{n-1}$ have a (non-vanishing) conductivity σ_n. Given a time-dependent magnetic source situated at a height h above the plane surface ($z=0$) of the conductor and assuming that the intervening region $-h < z < 0$ is free space, determine the total electromagnetic field comprising the field of the currents induced in the conductor and the field of the source.

In practice, of course, the geophysicist is usually concerned with the inverse problem of having to infer the layer conductivities, and occasionally the source field, from an analysis of magnetic field variations, supplemented in some cases by telluric measurements, recorded at one or several localised stations on the earth's surface. The various procedures that have been devised for doing this, such as the magnetotelluric method, geomagnetic deep sounding and the electromagnetic method using an artificial source, are really separate subjects in their own right but all of them depend ultimately on the basic problem of electromagnetic induction in the layered earth, outlined above.

Much of the early work in induction theory was developed with world-wide effects in mind so that the earth was represented by a conducting sphere and the field was expressed as a series of spherical harmonics. This approach precluded, however, an examination of those strictly local effects for which the field could only be described by many harmonics of high order, and for which, therefore, the earth is best treated as a conducting half space with a plane surface. It was not until Price (1950) published what has become the classic paper in the field that a general theory of electromagnetic induction in a conductor of this type became available. Although he considered only a non-layered homogeneous conductor, his theory was otherwise both comprehensive and general, and provided a complete explanation of all the physical processes involved.

Price developed his general theory by considering the elementary solutions obtained by separating the variables in the differential equation satisfied by the field vectors. Two distinct types of solutions emerged. Those of the "first type" corresponded to current systems flowing inside the conductor parallel to its surface, and possessing a magnetic field outside the

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conductor. Such currents are either induced by an external magnetic source or are freely decaying from some initial distribution. The solutions of the "second type" corresponded solely to freely decaying current systems having no external magnetic field. In induction problems, the systems of free decay are unimportant (indeed non-existent if we assume the initial distribution of currents in the conductor to be zero), so that only solutions of the first type are required. These and the other principal features of Price's theory have been summarised by Rikitake (1966), and also by Price (1967) himself in a review of electromagnetic induction within the earth.

In a sequel to Price's paper, Gordon (1951b) presented a rather different approach to the general theory. He considered induction by both magnetic and electric sources and obtained solutions by constructing an analogous problem in heat conduction. More recently Weaver (1971a) noted that if the field were represented by magnetic and electric Hertz vectors directed normally to the surface of the conductor, then the solutions associated with these vectors corresponded to Price's solutions of the first and second type, respectively. Thus, he was able to reformulate the theory of induction (as distinct from the free decay of currents) quite concisely in terms of just the one scalar component of the magnetic Hertz vector, and obtained general soltuions for the electromagnetic field vectors by relating them through simple formulas to the double Fourier transform in x and y of the magnetic Hertz potential of the source evaluated at the surface of the conductor.

The more general problem of electromagnetic induction in a multi-layered earth has been considered by several authors, at least for periodic sources. Price (1962), in a discussion of the magnetotelluric method, indicated the extension of his theory to a two-layer earth, and this was further extended to the N-layer case by Srivastava (1965). An elegant treatment of induction in a multi-layered conductor has also been given by Schmucker (1970), while Summers and Weaver (1973) have generalised Weaver's version of the theory for a uniform earth.

In all the references cited so far, displacement currents were neglected at the outset and the induction theory was derived from the quasi-static form of Maxwell's equations, the approximation being justified by the fact that in geophysical applications the time of electromagnetic propagation across the region of interest is negligible compared with the time scale of the field variations. An alternative procedure is to develop an exact theory and then to extract the properties of the induction field by making a near-field approximation in the electromagnetic wave solutions. This approach, however, is usually more complicated and suffers the disadvantage that it conceals the main physical features of the induction field behind a more elaborate mathematical structure than is really necessary. Indeed, Price (1962) and Bullard and Parker (1970) have cautioned against drawing misleading conclusions from the physical picture of real electromagnetic wave propagation. Nevertheless, some authors prefer the greater generality of the exact theory, notably Ward (1967) who has developed for geophysical application, the fields of a variety of sources over a multi-layered earth using a full-wave treatment. Although written mainly from the radio scientists' point of view, the books by Brekhovskikh (1960) and Wait (1970) also contain much pertinent information, and a useful handbook summarising many of the important formulas has been compiled by Kraichman (1970). Fournier (1966) has published a comprehensive bibliography of papers in electromagnetic induction which have contributed specifically to the development of the magnetotelluric methods.

2. Basic equations

The mathematical analysis will be developed in S.I.-units and we shall regard $\boldsymbol{E}$, the electric field intensity, and $\boldsymbol{B}$, the magnetic induction, as the fundamental field vectors, henceforth referring to them loosely as the electric and magnetic fields, respectively. It will be assumed that the permeability has the constant free-space value μ_0 everywhere. Variations in the permeability of the earth are just not important enough to warrant carrying the burden of this extra variable in the analysis.

Electromagnetic induction is governed by the equations:

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \quad (1)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \sigma \boldsymbol{E} \quad (2)$$

where σ is the conductivity of the medium, and the displacement current term has been neglected in eq. 2. Many authors prefer to use electromagnetic units for induction problems. Formally we can convert to this system by simply replacing μ_0 by 4π and understanding $\boldsymbol{B}$ to represent both the magnetic induction and the magnetic field intensity.

In the non-conducting region, eq. 2 reduces to $\nabla \times \boldsymbol{B} = 0$ which implies that the magnetic field can be expressed as the gradient of a scalar potential. The source of such a field is called a quasi-static *magnetic* source. On the other hand a quasi-static *electric* source for which $\nabla \times \boldsymbol{E} = 0$ (e.g., a slowly periodic electric dipole) does not properly belong to the first-order induction theory defined by eq. 1 and 2, for they show that the time-dependent magnetic field associated with such a source is necessarily vanishing so that there is no agent for inducing currents in the conducting medium. Eq. 2 must be modified to include second-order effects in the non-conducting region when electric sources are present. Gordon (1951b) has shown how this can be done in the general theory for a uniform earth. However, apart from the horizontal electric dipole, whose quasi-static field above a stratified conductor has been derived elsewhere (Bannister, 1966; Mundry, 1967; Vanyan, 1968), sources of electric type have a limited application in electromagnetic sounding and will not be considered further in this paper.

By introducing electric and magnetic Hertz vectors, $\Pi(\boldsymbol{r},z,t)\hat{\boldsymbol{z}}$ and $\Gamma(\boldsymbol{r},z,t)\hat{\boldsymbol{z}}$ respectively, where $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ denote unit vectors along the coordinate axes, we can express the field vectors in the form (Weaver, 1971a):

$$\boldsymbol{E} = \nabla[\Pi'(\boldsymbol{r},z,t)] - \frac{\partial}{\partial t}[\mu_0\sigma\Pi(\boldsymbol{r},z,t)\hat{\boldsymbol{z}} + \nabla\times\{\Gamma(\boldsymbol{r},z,t)\hat{\boldsymbol{z}}\}] \quad (3)$$

$$\boldsymbol{B} = \nabla[\Gamma'(\boldsymbol{r},z,t)] - \mu_0\sigma\left[\frac{\partial}{\partial t}\Gamma(\boldsymbol{r},z,t)\hat{\boldsymbol{z}} - \nabla\times\{\Pi(\boldsymbol{r},z,t)\hat{\boldsymbol{z}}\}\right] \quad (4)$$

(A prime on the function symbol denotes differentiation with respect to z.) Eq. 1 and 2 are then automatically satisfied provided that the Hertz potentials Π and Γ both satisfy the induction (diffusion) equation:

$$(\nabla^2 - \mu_0\sigma\partial/\partial t)\Phi(\boldsymbol{r},z,t) = 0 \quad (5)$$

where Φ is a scalar function which represents either Hertz potential. In a non-conductor eq. 5 reduces to Laplace's equation.

It will be seen by eq. 3 that the Γ-field has no electric z-component and by eq. 4 that the Π-field has no magnetic z-component (in fact no magnetic field at all in a non-conductor). For this reason their separate solutions are sometimes called the transverse magnetic and transverse electric modes, respectively; they correspond to Price's (1950) solutions of the first and second kind. It is also clear from eq. 3 that $-\Pi'$ is the scalar potential of the field and from eq. 4 that $-\Gamma'$ is the magnetic scalar potential in a non-conductor.

The induction problem is therefore a question of solving eq. 5 subject to the boundary conditions:

$$c_n\Phi(\boldsymbol{r},z_n-0,t) = c_{n+1}\Phi(\boldsymbol{r},z_n+0,t) \quad (6)$$

$$\Phi'(\boldsymbol{r},z_n-0,t) = \Phi'(\boldsymbol{r},z_n+0,t) \quad (7)$$

at each interface $z = z_n$ ($n=0,1,...,N-1$), with Φ representing Π when $c_0 = 0$, $c_n = \sigma_n$ ($n=1,2,...,N$), and Γ when $c_n = 1$ ($n=0,1,...,N$). That these are indeed the correct relations for the Hertz potentials is easily verified by applying the usual boundary conditions on $\boldsymbol{E}$ and $\boldsymbol{B}$ as defined in eq. 3 and 4.

3. Solution of the induction equation

With the understanding that $\sigma_0 = 0$, it is convenient to regard the region $-h < z < 0$ above the conductor as layer 0. Applying the Laplace transform:

$$\overline{\Phi}(\boldsymbol{r},z,s) = \int_0^\infty \Phi(\boldsymbol{r},z,t)\,\mathrm{e}^{-st}\,\mathrm{d}t \quad (8)$$

to eq. 5 for the potential in the n-th layer we obtain:

$$(\nabla^2 - \mu_0\sigma_n s)\overline{\Phi}(\boldsymbol{r},z,s) = 0 \quad (9)$$

where we have assumed that $\Phi(\boldsymbol{r},z,0) = 0$ inside the conductor. This assumption is permissible because a non-vanishing initial value of Φ leads only to an additional term in the solution whose time-dependence is an exponential decay factor. In other words, electric currents which are already flowing in the conductor at $t = 0$, decay in time along with their associated electromagnetic field quite independently of any external inducing field and are therefore unimportant when considering induction by an overhead source. These solutions have been discussed in more mathematical detail for a uniform earth by Price (1950). His "free modes of decay of the first type" correspond to the decay part of the solution for Γ while those of the "second type" correspond to the field associated with Π (Weaver, 1971a).

If a double Fourier transform with respect to $\boldsymbol{r}$, denoted in formal notation by:

$$\bar{F}(\boldsymbol{\rho},z,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\Phi}(\boldsymbol{r},z,s)\ e^{i\boldsymbol{r}\cdot\boldsymbol{\rho}}\ d\boldsymbol{r} \tag{10}$$

with $\boldsymbol{\rho} = \xi\hat{x} + \eta\hat{y}$, is now applied to eq. 9 we obtain the differential equation:

$$\bar{F}''(\boldsymbol{\rho},z,s) = \lambda_n^2 \bar{F}(\boldsymbol{\rho},z,s) \tag{11}$$

where:

$$\lambda_n = (\rho^2 + \mu_0\sigma_n s)^{\frac{1}{2}} \tag{12}$$

Subject to the boundary conditions, eq. 6 and 7, and to the vanishing of the field as $z \to \infty$ in the N-th layer, the solution of eq. 11 can be expressed in terms of the transformed field of the source denoted by $\bar{F}^s$. The analysis is formally the same as that given for periodic sources by Summers and Weaver (1973) with the angular frequency replaced by $-is$, and we shall quote their results directly.

Writing:

$$U_n = \lambda_n \operatorname{cosech} \lambda_n d_n\ , \qquad V_n = \lambda_n \operatorname{cotgh} \lambda_n d_n\ ,$$

and noting that $\lambda_0 \equiv \rho$ by definition 12, we define:

$$Q_0(\boldsymbol{\lambda}) = \frac{2\lambda_0}{\lambda_0 + V_1 - U_1 Q_1(\boldsymbol{\lambda})} \tag{13}$$

$$Q_{N-1}(\boldsymbol{\lambda}) = \frac{U_{N-1}}{V_{N-1} + \lambda_N} \tag{14}$$

and for $n = 1, 2, ..., N-2$:

$$Q_n(\boldsymbol{\lambda}) = \frac{U_n}{V_n + V_{n+1} - U_{n+1} Q_{n+1}(\boldsymbol{\lambda})} \tag{15}$$

where the vector argument $\boldsymbol{\lambda}$ indicates a functional dependence on the variables $(\lambda_0, \lambda_1, ..., \lambda_N)$. Then the solutions for the transforms of the separate electric and magnetic Hertz potentials, denoted by $\bar{P}$ and $\bar{G}$, respectively, are (a) in the region $-h < z < 0$:

$$\bar{P}(\boldsymbol{\rho},z,s) = \bar{P}^s(\boldsymbol{\rho},z,s) + \bar{P}^s(\boldsymbol{\rho},-z,s) \tag{16}$$

$$\bar{G}(\boldsymbol{\rho},z,s) = \bar{G}^s(\boldsymbol{\rho},z,s) - \bar{G}^s(\boldsymbol{\rho},-z,s) + \bar{G}_0(\boldsymbol{\rho},z,s) \tag{17}$$

and (b) in each layer $z_{n-1} < z < z_n$ $(n=1,2,...,N)$:

$$\bar{P}(\boldsymbol{\rho},z,s) = 0\ , \quad \bar{G}(\boldsymbol{\rho},z,s) = \bar{G}_n(\boldsymbol{\rho},z,s) \tag{18}$$

where for $n = 0, 1, ..., N$:

$$\bar{G}_n(\boldsymbol{\rho},z,s) = \bar{g}_n(\rho,z,s)\bar{G}^s(\boldsymbol{\rho},0,s) \tag{19}$$

with:

$$\bar{g}_0(\rho,z,s) = Q_0(\boldsymbol{\lambda})\ e^{\rho z} \tag{20}$$

$$\bar{g}_N(\rho,z,s) = \exp\left[\lambda_N(z_{N-1} - z)\right] \prod_{k=0}^{N-1} Q_k(\boldsymbol{\lambda}) \tag{21}$$

and for $n = 1, 2, ..., N-1$:

$$\bar{g}_n(\rho,z,s) = \left[\prod_{k=0}^{n-1} Q_k(\boldsymbol{\lambda})\right] \frac{Q_n(\boldsymbol{\lambda}) \sinh\left[\lambda_n(z - z_{n-1})\right] + \sinh\left[\lambda_n(z_n - z)\right]}{\sinh \lambda_n d_n} \tag{22}$$

Note that the function $\bar{g}_n(\rho,z,s)$ depends, through the variables λ_n, only on the *scalar* ρ, and also that since $\bar{F}^s(\boldsymbol{\rho},z,s)$ represents a transformed potential of a source located at $z = -h$, $\bar{F}^s(\boldsymbol{\rho},-z,s)$ is the potential of an identical but fictitious source at $z = +h$, i.e., the image of the real source reflected in the surface of the conductor.

Eq. 16, 17 and 18 can be Fourier-Laplace inverted, immediately to give the solutions for Π and Γ directly in terms of the Hertz potentials of the source and its image and the function:

$$\Gamma_n(\boldsymbol{r},z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t g_n(\rho,z,\tau)\ G^s(\boldsymbol{\rho},0,t-\tau)\ e^{-i\boldsymbol{\rho}\cdot\boldsymbol{r}}\ d\tau\ d\boldsymbol{\rho} \tag{23}$$

which is the result of inverting eq. 19 and using the convolution theorem for Laplace transforms. Since the source field is assumed to be given, and since the function g_n is determined through eq. 20, 21 and 22 by the known conductivity distribution of the conductor, this effectively completes the solution of the induction problem posed in section 1. The main obstacle to extracting actual solutions to a given problem lies in finding the correct expression for g_n by taking the inverse Laplace transform of $\bar{g}_n$, itself a very complicated function. Fortunately, this difficulty largely disappears in the case of periodic sources to be discussed later.

4. The electromagnetic field

It is convenient to consider separately the components of the field parallel to and normal to the surface of the conductor by writing:

$$E = E_{\parallel} + E_z \hat{z}, \quad B = B_{\parallel} + B_z \hat{z} \tag{24}$$

Let E^s, B^s denote the field vectors of the source and E^0, B^0 those of the image. Bearing in mind that a differentiation in z of the image potentials introduces a change in sign we deduce from eq. 3 and 4 and the inverted forms of eq. 16, 17 and 18 that (a) in the region $-h < z < 0$:

$$\left.\begin{aligned} E_{\parallel} &= E_{\parallel}^s - E_{\parallel}^0 - \nabla \times [\hat{z}\partial\Gamma_0(r,z,t)/\partial t] \\ E_z &= E_z^s + E_z^0 \end{aligned}\right| \tag{25}$$

$$\left.\begin{aligned} B_{\parallel} &= B_{\parallel}^s + B_{\parallel}^0 + \nabla_{\parallel}[\Gamma_0'(r,z,t)] \\ B_z &= B_z^s - B_z^0 + \Gamma_0''(r,z,t) \end{aligned}\right| \tag{26}$$

and (b) in the layer $z_{n-1} < z < z_n$ $(n=1,2,...,N)$:

$$\left.\begin{aligned} E_{\parallel} &= -\nabla \times [\hat{z}\partial\Gamma_n(r,z,t)/\partial t] \\ E_z &= 0 \end{aligned}\right| \tag{27}$$

$$\left.\begin{aligned} B_{\parallel} &= \nabla_{\parallel}[\Gamma_n'(r,z,t)] \\ B_z &= -\nabla_{\parallel}^2[\Gamma_n(r,z,t)] \end{aligned}\right| \tag{28}$$

The entire electromagnetic field has now been expressed in terms of the source and image fields and the magnetic Hertz potential defined in eq. 23. No reference to the electric Hertz potential is required. This is because the vertical component of the electric field, which is governed solely by the electric Hertz potential, is unimportant in the theory of induction for horizontally stratified conductors. It vanishes completely inside the conductor showing that all current flow is parallel to the surface. (Vertical currents can arise in the free modes of decay of the second type (Price, 1950) but then, of course, Π has a non-vanishing solution since its initial value can no longer be discarded.) If a vertical component of electric field is present in the region of free-space above the conductor it arises merely from the type of source present together with the surface charge it induces on the surface of the conductor. Vertical currents must flow inside the conductor in order to maintain this varying charge distribution but their magnitudes are of the order of displacement currents and their neglect is therefore quite consistent with the hypotheses of induction theory (Price, 1950, 1962).

5. Two- and one-layer earth models

In practice the recursion relation 15 defining the factors Q_n for a multi-layered earth must be solved numerically for each problem with its own set of data. When there are only two layers, however, simple algebraic expressions for the factors Q_0 and Q_1 (the only ones which exist when $N = 2$), viz.:

$$Q_0(\lambda) = \frac{2\rho(\lambda_1 + \lambda_2 \operatorname{tgh} \lambda_1 d_1)}{\lambda_1(\rho + \lambda_2) + (\lambda_1^2 + \rho\lambda_2) \operatorname{tgh} \lambda_1 d_1} \tag{29}$$

$$Q_1(\lambda) = \frac{2\lambda_1 \, e^{-\lambda_1 d_1}}{\lambda_1 + \lambda_2 + (\lambda_1 - \lambda_2)\, e^{-2\lambda_1 d_1}} \tag{30}$$

are obtained by putting $N = 2$ and $z_1 = d_1$, in eq. 14 and substituting in eq. 13. These results are worth recording because in many applications (seawater–seabed, crust–mantle, etc.) a two-layered model of the earth suffices.

The corresponding expressions for a homogeneous (one-layer) earth can be found by putting $\lambda_1 = \lambda_2 = \lambda$ in eq. 29 and 30. They are:

$$Q_0(\lambda) = 2\rho/(\rho + \lambda), \quad Q_1(\lambda) = e^{-\lambda d_1} \tag{31}$$

and their substitution in eq. 20, 21 and 22 (with $n = 1$ and $N = 2$) gives:

$$\bar{g}_0(\rho,z,s)\,e^{-(\rho+\lambda)z} = \bar{g}_1(\rho,z,s) = \bar{g}_2(\rho,z,s) \quad (32)$$

where:

$$\bar{g}_1(\rho,z,s) = 2\rho\, e^{-\lambda z}/(\rho+\lambda) \quad (33)$$

Recalling the dependence of λ on s in eq. 12, we can invert the Laplace transform (eq. 33) by making a simple change of variable in a tabulated result (Erdélyi, 1954), to obtain:

$$g_1(\rho,z,t) = \frac{2\rho}{\mu_0\sigma}\left[\frac{\beta}{\sqrt{\pi}}\exp\left\{\left(\frac{\rho}{\beta}\right)^2 + \left(\frac{\beta z}{2}\right)^2\right\} - \rho\, e^{\rho z}\,\mathrm{erfc}\left\{\frac{\rho}{\beta}+\frac{\beta z}{2}\right\}\right] \quad (34)$$

where $\beta = \sqrt{(\mu_0\sigma/t)}$, and it follows immediately from eq. 32 that:

$$g_0(\rho,z,t) = g_1(\rho,0,t)\,e^{\rho z} \quad (35)$$

6. Transient and periodic inducing fields

The assumption that initially there are no current systems inside the conductor implies that the source too is nonexistent prior to $t = 0$. Otherwise it would have already induced a current flow thereby contradicting the assumed initial conditions. Let us suppose, therefore, that a periodic inducing source of angular frequency ω is suddenly created at the instant $t = 0$, i.e.:

$$\Gamma^s(r,z,t) = H(t)\,e^{i\omega t}\,\Gamma^s(r,z,0) \quad (36)$$

where $H(t)$ is the Heaviside function with the value 1 for $t > 0$ and 0 for $t < 0$. Substituting eq. 36 in eq. 23 and rearranging the time integral by applying the definition of the Laplace transform in eq. 8, we obtain:

$$\Gamma_n(r,z,t) = \frac{e^{i\omega t}}{2\pi}\int_{-\infty}^{\infty} G^s(\boldsymbol{\rho},0,0)\,e^{-i\boldsymbol{\rho}\cdot\boldsymbol{r}}\left[\bar{g}_n(\rho,z,i\omega) - \int_t^{\infty} g_n(\rho,z,\tau)\,d\tau\right]d\boldsymbol{\rho} \cdot (37)$$

Two important types of time-dependence can be extracted from eq. 36. The first is obtained by taking $\omega = 0$, whence:

$$\Gamma^s(r,z,t) = H(t)\Gamma^s(r,z,0). \quad (38)$$

which represents an aperiodic source that instantaneously produces an external, static magnetic field. This type of source is used in transient electromagnetic sounding of the earth (Keller and Frischknecht, 1966; Vanyan, 1967; Keller, 1971). After the substitution $\omega = 0$ in eq. 37 it is a formidable problem to proceed any further analytically unless the earth is assumed to be homogeneous. In that case g_1 is given by eq. 34 which, despite its complicated form, can be integrated to give:

$$\int_t^{\infty} g_1(\rho,z,\tau)\,d\tau = \chi(\rho/\beta, z\beta) \quad (39)$$

where:

$$\chi(u,v) = \tfrac{1}{2}\,e^{-uv}\,\mathrm{erfc}(u-\tfrac{1}{2}v) + e^{uv}(\tfrac{1}{2}+uv+2u^2)\,\mathrm{erfc}(u+\tfrac{1}{2}v) - (2u/\sqrt{\pi})\exp(-u^2-v^2) \quad (40)$$

It also follows from eq. 32, 33 and 12 that:

$$\bar{g}_0(\rho,z,0) = e^{\rho z}, \quad \bar{g}_1(\rho,z,0) = e^{-\rho z} \quad (41)$$

Now the transformed source potential $\bar{G}^s$ satisfies eq. 11 (with $\lambda_0 \equiv \rho$) and vanishes as $z \to \infty$, so that its inverse Laplace transform has the solution:

$$G^s(\boldsymbol{\rho},z,t) = G^s(\boldsymbol{\rho},0,t)\,e^{-\rho z} \quad (42)$$

Hence eq. 37 with $\omega = 0$ and $n = 1$ becomes:

$$\Gamma_1(r,z,t) = \Gamma^s(r,z,0) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\rho/\beta, z\beta)\,G^s(\boldsymbol{\rho},0,0)\,e^{-i\boldsymbol{\rho}\cdot\boldsymbol{r}}\,d\boldsymbol{\rho} \quad (43)$$

This gives the transient field inside the conductor for $t > 0$. Likewise, by substituting the formulas 35 and 41 in eq. 37 with $\omega = 0$ and $n = 0$, and using eq. 42 again with z replaced by $-z$ we find that:

$$\Gamma_0(r,z,t) = \Gamma^s(r,-z,0) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\rho/\beta,0)\,G^s(\boldsymbol{\rho}\,0,0)\exp(\rho z - i\boldsymbol{\rho}\cdot\boldsymbol{r})\,d\boldsymbol{\rho} \quad (44)$$

When this result is substituted in eq. 25 and 26 to get the transient electromagnetic field in the region $-h < z < 0$, the image potential in eq. 44 leads to terms which exactly cancel the other image terms in the field expressions. The results in eq. 43 and 44 are special forms, corresponding to unit-relative permeability, of general solutions derived by Weaver (1971a) for induction in a uniformly magnetic and conducting half-space by the aperiodic source defined by eq. 38.

The other important example included in eq. 36 is its asymptotic form as $t \to \infty$ which gives the simple harmonic source:

$$\Gamma^s(r,z,t) \sim e^{i\omega t}\,\Gamma^s(r,z,0) \tag{45}$$

In the corresponding asymptotic representation of eq. 37 the transient part of the solution dies out leaving the steady-state periodic field:

$$\Gamma_n(r,z,t) \sim \frac{e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} G^s(\boldsymbol{\rho},0,0)\, \bar{g}_n(\rho,z,i\omega)\, e^{-i\boldsymbol{\rho}\cdot\boldsymbol{r}}\, d\boldsymbol{\rho} \tag{46}$$

Here $\bar{g}_n(\rho,z,i\omega)$ is defined by eq. 20, 21 and 22 in which, according to eq. 12, the variables λ_n are replaced by:

$$\gamma_n = (\rho^2+\alpha_n^2)^{1/2} \tag{47}$$

where:

$$\alpha_n = (i\omega\mu_0\sigma_n)^{1/2} = (1+i)/\delta_n \tag{48}$$

δ_n denoting the skin depth of the n-th layer.

7. The complex image

Even though the general solution for a periodic field has been expressed by eq. 46 in a relatively simple form, it still involves a closed Fourier integral which has to be evaluated numerically for each specific problem. However, when only the field in the region $-h < z < 0$ is required, and this is invariably the case in geophysical applications, it is often possible to make a simple approximation which effectively eliminates the integrals. The idea seems to have originated with the intuitive observation of Ball et al. (1966) that if the image of a quasi-static source at height h above a homogeneous earth is imagined to be located at the *complex* depth $h + \zeta$, where:

$$\zeta = \delta(1-i) \tag{49}$$

and δ is the skin depth defined in eq. 48, then the combined field of source and image gives a good approximation to the total field on and above the earth's surface. Bannister (1968a, 1969, 1970) demonstrated the general validity of this concept by comparing calculations based on the image formulas with those obtained from the exact solutions for a variety of sources.

A formal analytical justification of applying the technique to vertical magnetic-dipole and horizontal-line current sources was provided by Wait (1969) and Wait and Spies (1969), respectively. Their analyses indicated that the approximation was valid to the extent that third-order spatial derivatives of the image field were negligible, i.e., at distances somewhat greater than δ from the ordinary (real) image of the source. Weaver (1971b) extended the theory to apply to an arbitrary magnetic source and showed that the approximate forms of eq. 25 and 26 for the field vectors in $-h < z < 0$ were:

$$\boldsymbol{E}_{\parallel} \approx \boldsymbol{E}_{\parallel}^{s} - \boldsymbol{E}_{\parallel}^{*} + \boldsymbol{\nabla}_{\parallel}\left[\int_{-\infty}^{z} (E_z^0 - E_z^*)\, dz\right] \tag{50}$$

$$\boldsymbol{B}_{\parallel} \approx \boldsymbol{B}_{\parallel}^{s} + \boldsymbol{B}_{\parallel}^{*} \quad \text{and} \quad B_z \approx B_z^s - B_z^* \tag{51}$$

where $\boldsymbol{E}^*$ and $\boldsymbol{B}^*$ are the image fields obtained by replacing z by $\zeta - z$ in the corresponding expressions for the source. The exact formula for E_z in eq. 25 remains, of course, unchanged. Eq. 50 and 51 represent a considerable simplification of the exact solutions, containing closed integrals. Only eq. 50 offers any complication, but if $E_z^s = 0$, even it reduces to a very simple form.

The technique applies also to an N-layer earth. It can be shown that the appropriate expression for ζ in this case is:

$$\zeta = \lim_{\rho \to 0} \frac{Q_0(\boldsymbol{\gamma})}{\rho} = \frac{(1-i)\delta_1 \sinh \alpha_1 d_1}{\cosh \alpha_1 d_1 - Q_1(\boldsymbol{\alpha})} \tag{52}$$

where the vector arguments $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ now comprise the $(N+1)$ variables γ_n and α_n ($n=0,1,...,N$), respectively, defined in eq. 47 and 48, and that a sufficient condition for the approximate formulas 50 and 51 to hold for a source at $r = 0$, $z = -h$, is:

$$[r^2+(h+z)^2]^{\frac{3}{2}} \gg \cdot |\zeta|^3 \tag{53}$$

A formal proof of these results will be published elsewhere.

When $N = 2$, Q_1 is given by eq. 30 so that eq. 52 reduces to:

$$\zeta = (1-i)\delta_1 \frac{1 + \kappa\, e^{-2\alpha_1 d_1}}{1 - \kappa\, e^{-2\alpha_1 d_1}} \tag{54}$$

where $\kappa = [1-(\sigma_2/\sigma_1)^{\frac{1}{2}}]/[1+(\sigma_2/\sigma_1)^{\frac{1}{2}}]$, a result obtained by Thomson and Weaver (1970) and applied by them to the field of a vertical magnetic dipole over a two-layer earth. The simple formula 49 which was the one originally proposed for a homogeneous earth can be recovered from eq. 54 by putting $\sigma_1 = \sigma_2 = \sigma$.

8. Dipole sources

Current loops are widely used as artificial sources for electromagnetic sounding of the earth. If the loop is small it may, of course, be regarded as a magnetic dipole whose moment $\boldsymbol{M}(t)$ is directed normally to the plane of the loop, and for this reason induction by a magnetic dipole has been the subject of numerous investigations in the geophysical literature. The book by Baños (1966) presents an exhaustive analysis of dipole radiation fields in the presence of a uniform conducting half-space, and the article by Ward (1967) contains a general treatment for geophysicists of dipole fields over a layered earth.

The quasi-static electromagnetic field of a magnetic dipole located at $r = 0$, $z = -h$ is well known to be:

$$\boldsymbol{E}^{s} = \boldsymbol{R} \times \boldsymbol{M}'(t)/4\pi R^3 \tag{55}$$

$$\boldsymbol{B}^{s} = -\nabla[\boldsymbol{R}\cdot\boldsymbol{M}(t)/4\pi R^3] \tag{56}$$

where $\boldsymbol{R} = \boldsymbol{r} + (h+z)\hat{\boldsymbol{z}}$ is the position vector from the dipole. Following the remarks made in section 2 we can identify the function $-\Gamma^{s\prime}$ at once as the magnetic scalar potential in eq. 56. Hence by integration and Fourier transformation (Weaver, 1971a):

$$G^{s}(\boldsymbol{\rho},0,t) = (i\boldsymbol{\rho} + \rho\hat{\boldsymbol{z}}) \cdot \boldsymbol{M}(t)/4\pi\rho^2 \tag{57}$$

Eq. 55 and 56, and eq. 58 substituted in eq. 23, together define the total electromagnetic field in all regions according to the general solutions 25, 26, 27 and 28.

The moment of a vertical dipole aligned in the z-direction satisfies $\boldsymbol{\rho}\cdot\boldsymbol{M}(t) = 0$. This makes G^s in eq. 57 dependent only on the scalar ρ so that the double Fourier integral 23 expressed in polar coordinates can be immediately integrated with respect to its angular variable by Bessel's integral of order zero. Hence Γ_n becomes a function of r rather than $\boldsymbol{r}$, a simplification resulting from the cylindrical symmetry of the configuration. The periodic and transient fields of a vertical dipole over a homogeneous earth have been analysed by Gordon (1951a) and Bhattacharyya (1959a,b; 1963), respectively, and a unified treatment of both types of field has been developed in great detail by Meyer (1962). Numerous investigations of the periodic field of a vertical dipole over a two-layer earth have been made (Wait, 1951, 1955, 1958a; Bhattacharyya, 1955; Schlichter and Knopoff, 1959; Quon, 1963; Frischknecht, 1967; Patra, 1970; Weaver and Thomson, 1970), but the more difficult analysis of the transient fields has received less attention. Lowndes (1957) has obtained approximate solutions for a vertical dipole with both step-like and impulsive time variations resting on the surface of an earth in which the conductivities of the two layers are nearly equal; the solutions for an elevated vertical dipole with an impulsive time variation have been estimated asymptotically for small t by Wait (1972); and Morrison et al. (1969) have computed the response of two- and three-layer earth models to a horizontal loop (vertical dipole) excited by half-sine wave pulses of current. The vertical dipole over a stratified earth of more than two layers has been examined by Kozulin (1960), Bannister (1966) and Ryu et al. (1970); and Wait (1962a) has analysed the response of a continuously stratified earth, which is equivalent to letting $N \to \infty$ and all $d_n \to 0$ in the layered model.

If the dipole is oriented horizontally with its moment parallel to the earth's surface we can put $\hat{\boldsymbol{z}} \cdot \boldsymbol{M}(t) = 0$ in eq. 57 and simplify eq. 23 again by using Bessel's integral of order one. In this case, however, there is no cylindrical symmetry. The fields of a horizontal dipole have been calculated by Wait (1953a, 1956) for the dipole on a uniform earth, and by Wait (1958a), Quon (1963) and Frischknecht (1967) for the dipole above a two-layer earth.

9. Electrojet sources

Non-uniform sources can also arise naturally in the form of the auroral and equatorial electrojets which are represented simply by a line current $I(t)\hat{y}$ flowing along the line $x = 0, z = -h$. In this case we have $E_z = 0$ and:

$$E^s - E^0 = \frac{\mu_0 I'(t)\hat{y}}{4\pi} \log \frac{x^2 + (h+z)^2}{x^2 + (h-z)^2} \tag{58}$$

$$B^s = \frac{\mu_0 I(t)}{4\pi} \nabla \operatorname{arctg} \frac{x}{h+z} \tag{59}$$

Integration and Fourier transformation of the magnetic scalar potential in eq. 59 (Weaver, 1971a) shows that:

$$G^s(\boldsymbol{\rho},0,t) = -i\mu_0 I(t)\, e^{-h\xi}\, \delta(\eta)/\xi|\xi| \tag{60}$$

Eq. 58, 59 and 60 determine the total field everywhere with the Dirac generalised function $\delta(\eta)$ in eq. 60 effectively reducing the double Fourier integral (eq. 23) to a single integral in ξ.

Price (1950) considered this source as an illustration of the application of his general theory for a uniform earth to a specific example. Law and Fannin (1961) approached the same problem using radiation fields although their final computations were for low-frequency fields of geophysical interest. Dosso (1966) presented further calculations based on their analysis. The line current above a stratified earth has been treated by several authors (Wait, 1953b, 1958b, 1970; Ward, 1967; Bannister, 1968b; Guldberg and Brock-Nannestad, 1970).

The analysis of this problem has also arisen in the application of the magnetotelluric method in equatorial and auroral zones (Hermance and Garland, 1968; Hermance and Peltier, 1970). For a more realistic representation of an electrojet, Peltier and Hermance (1971) considered a Gaussian distribution of line currents, and computed the corresponding magnetotelluric curves for a three-layer earth.

10. Elementary harmonic sources

Consider the non-uniform periodic source whose magnetic field is of the form:

$$B^s = -\nabla[\nu^{-1}\Lambda(\boldsymbol{r},\nu)\exp(i\omega t - \nu z)] \tag{61}$$

where $\nu > 0$. Its electric field is given by $E_z = 0$ and:

$$E^s = -(i\omega/\nu^2)\exp(i\omega t - \nu z)\,\nabla \times [\hat{z}\Lambda(\boldsymbol{r},\nu)] \tag{62}$$

Since $\nabla \cdot B^s = 0$, Λ must satisfy:

$$(\nabla_{\parallel}^2 + \nu^2)\Lambda(\boldsymbol{r},\nu) = 0 \tag{63}$$

We can find Γ^s in the usual manner by integrating the magnetic scalar potential of eq. 61 with respect to z. Then, by Fourier transformation, we obtain:

$$G^s(\boldsymbol{\rho},0,0) = L(\boldsymbol{\rho},\nu)/\nu^2 \tag{64}$$

where L is the Fourier transform of Λ, which according to eq. 63 satisfies:

$$(\rho^2 - \nu^2)L(\boldsymbol{\rho},\nu) = 0 \tag{65}$$

Interpreted as a generalised function (Jones, 1966), the solution of eq. 65 is:

$$L(\boldsymbol{\rho},\nu) = \underline{A}(\xi,\nu)\delta(\eta + \sqrt{\nu^2 - \xi^2}) + \underline{C}(\xi,\nu)\delta(\eta - \sqrt{\nu^2 - \xi^2}) \tag{66}$$

for $|\xi| < \nu$, and $L(\boldsymbol{\rho},\nu) = 0$ for $|\xi| > \nu$. When this solution is Fourier-inverted it reduces to a single integral because of the delta functions. Substituting $\xi = -\nu \cos\psi$ and writing $\boldsymbol{\nu} = \nu\cos\psi\hat{x} + \nu\sin\psi\hat{y}$ we can express the final solution as:

$$\Lambda(\boldsymbol{r},\nu) = \int_{-\pi}^{\pi} C(\boldsymbol{\nu})\, e^{i\boldsymbol{\nu}\cdot\boldsymbol{r}}\, d\psi \tag{67}$$

where $C(\boldsymbol{\nu}) = \dfrac{\nu\sin\psi}{2\pi}\begin{cases} \underline{A}(-\nu\cos\psi,\nu) & 0 \leqslant \psi < \pi \\ \underline{C}(-\nu\cos\psi,\nu) & -\pi < \psi \leqslant 0 \end{cases}$

By inserting eq. 64 and 66 into eq. 46 and simplifying the integral to the form of eq. 67 we obtain:

$$\Gamma_n(\boldsymbol{r},z,t) = \nu^{-2}\bar{g}_n(\nu,z,i\omega)\Lambda(\boldsymbol{r},\nu)\, e^{i\omega t} \tag{68}$$

Hence, by eq. 25 and 26 the total field in the region $-h < z < 0$ is:

$$E = i\omega\nu^{-2}\, e^{i\omega t}\,[2\sinh\nu z - \bar{g}_0(\nu,z,i\omega)]\, \nabla \times [\hat{z}\Lambda(\boldsymbol{r},\nu)] \tag{69}$$

$$B_{\parallel} = -\nu^{-1}\, e^{i\omega t}\,[2\cosh\nu z - \bar{g}_0(\nu,z,i\omega)]\, \nabla_{\parallel}[\Lambda(\boldsymbol{r},\nu)] \tag{70}$$

$$B_z = -e^{i\omega t}\,[2\sinh\nu z - \bar{g}_0(\nu,z,i\omega)]\,\Lambda(\boldsymbol{r},\nu) \tag{71}$$

The relation $\bar{g}_0' = \nu\bar{g}_0$, an immediate consequence of eq. 20, has been used in eq. 70 and 71. Likewise, by eq. 27 and 28 the field in the n-th layer $z_{n-1} < z < z_n$ is:

$$E = -i\omega\nu^{-2}\, e^{i\omega t}\, g_n(\nu,z,i\omega)\nabla\times[\hat{z}\Lambda(\mathbf{r},\nu)] \quad (72)$$

$$B_{\parallel} = \nu^{-2}\, e^{i\omega t}\, g_n'(\nu,z,i\omega)\nabla_{\parallel}[\Lambda(\mathbf{r},\nu)] \quad (73)$$

$$B_z = e^{i\omega t}\, g_n(\nu,z,i\omega)\Lambda(\mathbf{r},\nu) \quad (74)$$

the last result following from eq. 63. Expressions for $g_n(\nu,z,i\omega)$ are given by eq. 20, 21 and 22 with the variables λ_n, $(n=0,1,\ldots,N)$, now replaced by:

$$\theta_n = (\nu^2+\alpha_n^2)^{\frac{1}{2}} \quad (75)$$

where α_n is defined in eq. 48. It is clear that $E_z = 0$ and $\mathbf{E}\cdot\mathbf{B} = 0$ everywhere. The orthogonality of the electric and magnetic fields is a property of the particular inducing source in eq. 61. It does not hold in general (e.g., for the dipole field of section 8) but may be true for certain other sources such as the line current considered in section 9. Wait (1954) and Price (1962) have emphasized this fact, pointing out that a non-orthogonality of horizontal electric and magnetic fields does not necessarily imply an anisotropic conductivity of the earth.

Price (1950, 1962) developed his general theory in terms of the elementary solutions derived above, with $\Lambda(\mathbf{r},\nu)$ denoted by $\nu A(\nu)P(x,y,\nu)$ in his (1962) notation. Schmucker (1970) on the other hand, based his discussion on the solutions corresponding to:

$$\Lambda(\mathbf{r},\nu) = \nu A_0(\nu)\, e^{i\boldsymbol{\nu}\cdot\mathbf{r}} \quad (76)$$

which is a special case of eq. 67 obtained by writing $\boldsymbol{\nu} = \nu\cos\psi_0\hat{x} + \nu\sin\psi_0\hat{y}$ and choosing $C(\boldsymbol{\nu}) = \nu A_0(\nu)\delta(\psi-\psi_0)$. The parameter ν is seen by eq. 67 or eq. 76 to be a measure of the reciprocal of the horizontal "wavelength" of the source field. It characterises, therefore, the non-uniformity of the source. Price (1962) estimated that for geophysical applications involving sources of natural origin its values would lie in the range $1.57\cdot10^{-7}\ \mathrm{m}^{-1}$ to $1.57\cdot10^{-5}\ \mathrm{m}^{-1}$. As $\nu\to0$ the wavelength becomes infinite, the dependence of Λ on $\mathbf{r}$ disappears, and the magnetic field of the source becomes uniform. Uniform sources are discussed in section 11.

Elementary solutions are particularly useful for discussing the ratios of field components. The impedance at the depth z in the n-th layer is defined as $i\omega\mu_0 Z_n(z)$ where:

$$Z_n(z) = \frac{E_x}{i\omega B_y} = -\frac{E_y}{i\omega B_x} = -\frac{g_n(\nu,z,i\omega)}{g_n'(\nu,z,i\omega)} \quad (77)$$

by eq. 72 and 73. From eq. 21 and 22 it follows that:

$$\theta_N Z_N(z) = 1 \quad (78)$$

and for $n = 1, 2, \ldots, N-1$:

$$\theta_n Z_n(z) = \frac{\sinh[\theta_n(z_n-z)] + Q_n(\boldsymbol{\theta})\sinh[\theta_n(z-z_{n-1})]}{\cosh[\theta_n(z_n-z)] - Q_n(\boldsymbol{\theta})\cosh[\theta_n(z-z_{n-1})]} \quad (79)$$

At the surface the impedance is given by:

$$Z_1(0) = \frac{\sinh\theta_1 d_1}{\theta_1[\cosh\theta_1 d_1 - Q_1(\boldsymbol{\theta})]} \quad (80)$$

If the earth is uniform, Q_1 is given by eq. 31, and eq. 80 reduces to:

$$Z(0) = 1/\theta \quad (81)$$

Surface impedance is the fundamental measurement in the magnetotelluric method. For a given earth model, the surface impedance can be calculated from eq. 80 by the recursion relations 14 and 15. However, if they are substituted in eq. 80 directly, some algebraic rearrangement yields a recursion relation (which starts with eq. 78) for the impedance itself:

$$\theta_n Z_n(z_{n-1}) = \frac{\mathrm{tgh}(\theta_n d_n) + \theta_n Z_{n+1}(z_n)}{1 + \theta_n\, \mathrm{tgh}(\theta_n d_n) Z_{n+1}(z_n)} \quad (82)$$

a result which has been obtained by Wait (1953c, 1962b), Tikhonov and Shakhsuvarov (1956), and Schmucker (1970). An alternative way of writing eq. 79 is:

$$\theta_n Z_n(z) = \frac{e^{-\theta_n z} + D_n e^{\theta_n z}}{e^{-\theta_n z} - D_n e^{\theta_n z}} \quad (83)$$

where $D_N = 0$, and for $n = 1, 2, \ldots, N-1$:

$$D_n = \frac{Q_n(\boldsymbol{\theta})\, e^{-\theta_n z_{n-1}} - e^{-\theta_n z_n}}{e^{\theta_n z_n} - Q_n(\boldsymbol{\theta})\, e^{\theta_n z_{n-1}}} \quad (84)$$

Srivastava (1965) expressed eq. 83 in the form:

$$\theta_n Z_n(z) = -\mathrm{ctgh}(\theta_n z - \tfrac{1}{2}\log D_n) \quad (85)$$

and using the continuity of impedance at each interface $z = z_n$ he obtained the recursion relation:

$$\theta_n Z_n(z_{n-1}) = -\mathrm{cotgh}\,[\mathrm{arcotgh}\,\{-\theta_n Z_{n+1}(z_n)\} - \theta_n z_n] \quad (86)$$

which can be used instead of eq. 82 to generate the surface impedance from eq. 78. It is similar in form to the recursion relation derived earlier by the Russian workers and quoted by Berdichevsky (1960), and Miecznik (1966). By eq. 83 the surface impedance can also be written as:

$$Z_1(0) = \frac{1+D_1}{\theta_1(1-D_1)} \tag{87}$$

with D_1 obtained from $D_N = 0$ by the recursion relation for $n = 1, 2, ..., N-1$:

$$D_n = \frac{\Theta_+ D_{n+1}\, e^{z_n\Theta_-} - \Theta_-\, e^{-z_n\Theta_+}}{\Theta_+\, e^{-z_n\Theta_-} - \Theta_- D_{n+1}\, e^{z_n\Theta_+}} \tag{88}$$

where $\Theta_\pm = (\theta_{n+1} \pm \theta_n)$, which is readily verified from eq. 15 and 84. This formula was derived by Hutton (1972) although she actually expressed it in terms of the parameter $1/D_n$.

The recursion relations 82, 86 and 88 and all others (e.g., Nabetani and Rankin, 1969) giving the surface impedance on a layered earth are essentially equivalent. They have been used by many of the cited authors to compute magnetotelluric curves for non-uniform sources. The ratios B_z/B_x and B_z/B_y used in geomagnetic deep sounding can also be obtained from $Z_n(z)$ by multiplying it by the factors $i\omega E_x/B_z$ and $-i\omega E_y/B_z$, respectively. It is clear from eq. 72 and 74 that these latter ratios depend only on the source field. They are independent of g_n and hence of the conductivity distribution.

The other ratios of interest, which describe the attenuation of the field components within the n-th layer, are:

$$S_n(z) = \frac{E_x}{[E_x]_{z_{n-1}}} = \frac{E_y}{[E_y]_{z_{n-1}}} = \frac{B_z}{[B_z]_{z_{n-1}}} = \frac{g_n(\nu,z,i\omega)}{g_n(\nu,z_{n-1},i\omega)} \tag{89}$$

$$T_n(z) = \frac{B_x}{[B_x]_{z_{n-1}}} = \frac{B_y}{[B_y]_{z_{n-1}}} = \frac{g_n'(\nu,z,i\omega)}{g_n'(\nu,z_{n-1},i\omega)} \tag{90}$$

By eq. 21 and 22 they can be expressed as:

$$S_N(z) = T_N(z) = \exp[\theta_N(z_{N-1}-z)] \tag{91}$$

and for $n = 1, 2, ..., N-1$:

$$S_n(z) = \frac{Q_n(\boldsymbol{\theta})\sinh[\theta_n(z-z_{n-1})] + \sinh[\theta_n(z_n-z)]}{\sinh\theta_n d_n} \tag{92}$$

$$T_n(z) = \frac{Q_n(\boldsymbol{\theta})\cosh[\theta_n(z-z_{n-1})] - \cosh[\theta_n(z_n-z)]}{Q_n(\boldsymbol{\theta}) - \cosh\theta_n d_n} \tag{93}$$

These results were obtained by Schmucker (1970) in a slightly different form. Note that $S_n(z_n) = Q_n(\boldsymbol{\theta})$. This gives the physical interpretation of Q_n ($n=1,2,...,N-1$) as the ratio of the electric or vertical magnetic components at the bottom of the n-th layer to those at the top of the layer. Because of the continuity of the field components at each interface, the field at any depth can be related to its surface value through such identities as:

$$B_z/[B_z]_{z=0} = S_n(z)S_{n-1}(z_{n-1})....S_1(z_1) \tag{94}$$

In the unbounded N-th layer all the field components undergo the same exponential attenuation in eq. 91, commonly called the skin effect. In the layers of finite thickness, however, the skin effect does not apply and while E_x, E_y and B_z still attenuate together, the behavior of B_x and B_y is markedly different.

Finally it should be pointed out that all the preceding formulas involving the ratios of field components are strictly true only for the special type of inducing field assumed in eq. 61. When the general periodic solutions defined by eq. 46 are considered, the common factors which previously cancelled from numerator and denominator in the field ratios must now be retained since they appear as part of the integrands in a ratio of Fourier integrals. Clearly the condition for the simple ratio formulas to be approximately true in the general case is that the source function G^s must limit the values of $\boldsymbol{\rho}$ which contribute significantly to the integrals to a small bandwidth about some value $\boldsymbol{\nu}$. The relevant ratio may then be regarded as having the constant value corresponding to $\rho = \nu$ over the whole range of integration. The extent to which the approximation is valid will depend on the particular source under consideration.

11. Induction by a uniform field

The problem of electromagnetic induction in a plane earth by a uniform source is actually indeterminate. That is, for a given inducing magnetic field which is periodic in time but uniform in space, it is not possible to determine the induced field. Of course, the electromagnetic field within the earth can be expressed in terms of the *total* external magnetic field, but there is no way of uniquely separating this uniform field into its inducing and induced parts. Price (1950) clarified this point by considering the limiting case of a well-defined problem in spherical geometry in which the radius of the earth is a and the inducing field is defined by a spherical harmonic of degree m. He showed that when the earth's surface becomes plane by letting $a \to \infty$, the ratio of the tangential components of the induced and inducing fields tends to the value $m/(m+1)$. Now the field described by a surface harmonic of any finite degree can be regarded as essentially uniform over a sufficiently small portion of the spherical surface, so that in the limiting case as $a \to \infty$ a uniform inducing field is obtained, whatever the finite value of m. Thus the ratio of the induced to the inducing tangential parts of a uniform field over a plane earth can assume an infinite number of values between ½ and 1 for this particular limiting procedure. A non-uniform source over a plane earth is obtained by making $m \to \infty$ in such a way that $m/a \to \nu$, a finite limit, as $a \to \infty$.

Despite the indeterminacy of the problem, expressions for the ratios of the total field components can be found as in section 10. One way of doing this is to neglect displacement currents inside the earth only, and to consider plane electromagnetic waves normally incident on its surface. This was the procedure adopted by Cagniard (1953) and followed by many other authors. However, we have already remarked that the limiting case as $\nu \to 0$ of the results in section 10 correspond to a uniform inducing field, and this is clearly the simplest approach.

Now $\Lambda \to K_z$ and $\nu^{-1}\nabla_{\parallel}\Lambda \to -K_{\parallel}$ when we let $\nu \to 0$ in eq. 67, (or in eq. 76), where K_z and $K_{\parallel}$ are suitably defined constants, $K_{\parallel}$ being a vector parallel to the plane $z = 0$. Thus the limiting forms of eq. 61 and 62 are:

$$B^s = K\,e^{i\omega t}\,, \quad E^s - [E]_{z=0} = i\omega z\hat{z} \times K\,e^{i\omega t} \qquad (95)$$

with $K = K_{\parallel} + K_z\hat{z}$ representing the uniform magnetic field of the source. Since the formulas defining g_n all contain a factor Q_0 it follows that we can write:

$$\lim_{\nu\to 0}\,[\nu^{-1}\bar{g}_n(\nu,z,i\omega)] = \zeta f_n(z) \qquad (96)$$

where ζ is the complex length defined in eq. 52, and where f_n is a function which now satisfies:

$$f_n''(z) = \alpha_n^2 f_n(z) \qquad (97)$$

because $\bar{G}_n$, and hence $\bar{g}_n$, satisfy eq. 11. It is obvious from eq. 20 that $f_0(z) = 1$. The solutions for the total field (eq. 69–74) become therefore:

$$B = 2K_{\parallel}\,e^{i\omega t}\,, \quad E = i\omega(2z-\zeta)\hat{z} \times K\,e^{i\omega t} \qquad (98)$$

above the earth's surface, and:

$$\left.\begin{aligned} B &= -\zeta f_n'(z) K_{\parallel}\,e^{i\omega t}\,, \\ E &= -i\omega\zeta f_n(z)\hat{z} \times K\,e^{i\omega t} \end{aligned}\right| \qquad (99)$$

in the n-th conducting layer. Note that even though a uniform source may contain a vertical magnetic component, the total magnetic field is necessarily horizontal everywhere. In this example the total external magnetic field is double the horizontal magnetic component of the source which indicates that the ratio of the tangential components of the induced and inducing fields has its maximum value 1. This is not surprising since the fact that we started with a finite ν implies that we are dealing with the special case of a surface harmonic whose degree m has become infinite. Of course eq. 95 defines just one particular uniform source which gives rise to the solutions 98 and 99. The proportion of the total horizontal magnetic field which belongs to the source could, in fact, be any fraction between 0 and 1 without violating the boundary conditions at $z = 0$, and in addition the constant K_z is quite arbitrary since it is always nullified by an equal and opposite vertical component of the induced magnetic field.

It is clear from the definition (eq. 96) that a ratio of the functions f_n can be evaluated by taking the limit as $\nu \to 0$ of the corresponding ratio of functions g_n. Thus the impedance of a uniform field, given by eq. 99 in the form:

$$Z_n(z) = -f_n(z)/f_n'(z) \tag{100}$$

can be deduced immediately by letting $\nu \to 0$ in eq. 77, which is tantamount to replacing each θ_n by α_n in the formulas 78 and 79. This also applies to the other ratio formulas discussed in section 10. In particular, by virtue of eq. 52, eq. 80 becomes:

$$Z_1(0) = \tfrac{1}{2}\zeta \tag{101}$$

for a uniform source, and eq. 81 reduces to:

$$Z(0) = \tfrac{1}{2}(1-i)\delta \tag{102}$$

which is the well-known result of Cagniard (1953) for a homogeneous earth, and which led to his defining the "apparent conductivity" σ_a of a layered earth in terms of impedance measured at its surface by the formula:

$$\sigma_a^{-1} = \omega\mu_0|Z_1(0)|^2 \tag{103}$$

Differentiating eq. 100 and using eq. 97 we deduce that:

$$Z_n'(z) = [\alpha_n Z_n(z)]^2 - 1 \tag{104}$$

which is the Riccati equation. Eckhardt (1968) devised a simple graphical technique for computing the surface impedance of a layered earth by transforming eq. 104 into a differential equation satisfied by $\log[\alpha_n Z_n(z)]$, (which vanishes at $z = z_{n-1}$ by eq. 78), and plotting the characteristic solution paths. Thus by starting at $z = z_{n-1}$ and using the continuity of impedance at each interface one can use Eckhardt's chart to integrate upwards through the layers until the surface impedance is found. A different method has been given by Weidelt (1972). It is based on a formula he proved, that:

$$Z_1(0) = \lim_{z\to\infty} w(z)/W(z) \tag{105}$$

where w and W are solutions of the differential eq. 97 (regarded as applying to the whole earth with σ a function of z, rather than to a single layer) and subject to the boundary conditions:

$$w(0) = W'(0) = 0\,, \quad w'(0) = W(0) = 1 \tag{106}$$

This time one starts at the surface with the conditions 106 and integrates downwards until the ratio of the two solutions reaches a limiting value. Both Eckhardt's and Weidelt's methods may be used with a uniform source as alternative procedures to computing the surface impedance through the recursion relations discussed in section 10.

It remains to be decided under what circumstances the simplifying assumption of a uniform source is valid. Wait (1954) first considered this problem by comparing the surface impedance formulas for a uniform earth. Eq. 81 can be written as:

$$Z(0) = \tfrac{1}{2}(1-i)\delta(1-\tfrac{1}{2}i\nu^2\delta^2)^{-\frac{1}{2}} \tag{107}$$

so that eq. 102 is acceptable as an approximation provided that $(\nu\delta)^2 \ll 1$, i.e., that the horizontal scale of the inducing field is much less than a skin-depth. Later, Price (1962) pointed out that when the conductivity varies with depth, Wait's condition, while still necessary, may not be sufficient. The general impedance formula 80 depends on ν through every θ_n, $n = 1, 2, ..., N$, so that by eq. 75 Wait's condition must hold in every conducting layer if the neglect of ν is to be completely justified, i.e.:

$$\nu^2 \max\{\delta_n^2\} \ll 1 \tag{108}$$

In fact a less stringent condition will apply because no account has been taken of the varying degree to which conducting layers affect the field according to their depth beneath the surface. These questions have been considered more fully by Wait (1962b) and Niblett (1967).

12. Fields in a two-layer earth

We conclude this review by applying the field ratio formulas of section 10 to a two-layer earth, for which Q_1 is given by eq. 30, and for which therefore eq. 79, 92 and 93 give:

$$\theta_1 Z_1(0) = \frac{1 + \epsilon\, e^{-2\theta_1 d}}{1 - \epsilon\, e^{-2\theta_1 d}} \tag{109}$$

$$S_1(d_1) = \frac{(1+\epsilon)\, e^{-\theta_1 d}}{1 + \epsilon\, e^{-2\theta_1 d}} \tag{110}$$

$$T_1(d_1) = \frac{(1-\epsilon)\, e^{-\theta_1 d}}{1 - \epsilon\, e^{-2\theta_1 d}} \tag{111}$$

where $\epsilon = (\theta_1 - \theta_2)/(\theta_1 + \theta_2)$.

Consider first the case of a poorly conducting layer ($\nu^2\delta_1^2 \gg 1$) over a good conductor ($\nu^2\delta_2^2 \ll 1$) so

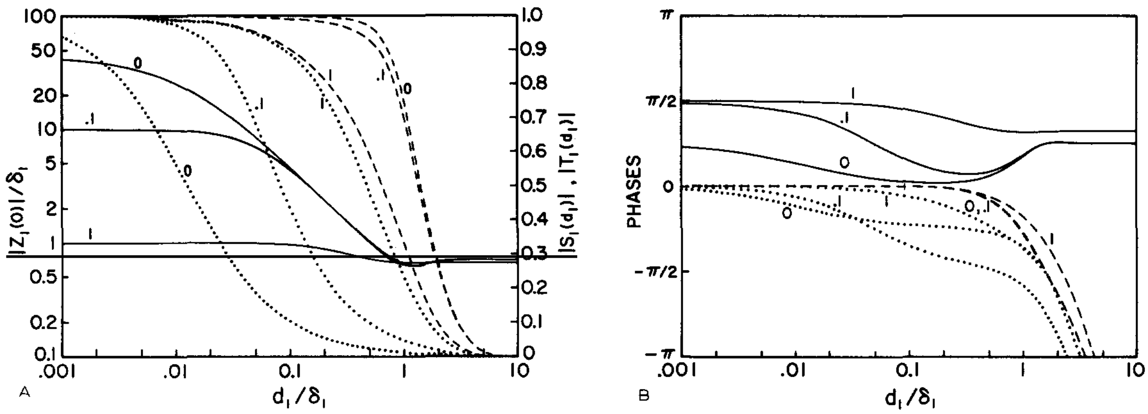

Fig. 1.A. Amplitudes (moduli) of the field ratios $Z_1(0)/\delta_1$ (solid line and left-side scale), $S_1(d_1)$ (broken line and right-side scale), and $T_1(d_1)$ (dotted line and right-side scale), plotted against the depth d_1/δ_1 (in skin depths) of the upper layer of a two-layer earth, for the parameters $\nu\delta_1$ = 0, 0.1 and 1. B. Phases (arguments) of the field ratios whose amplitudes are plotted in A.

that $\theta_1 \approx \nu$ and $\theta_2 \approx (1+i)/\delta_2$. We assume also that the depth of the upper layer is small compared with the dimensions of the source, i.e. $(\nu d_1)^2 \ll 1$. Then the first order approximation of eq. 109 is:

$$Z_1(0) = d_1 + \tfrac{1}{2}(1-i)\delta_2 \qquad (112)$$

Schmucker (1970) has used this formula to interpret the surface impedance in a way which provides more information than the apparent conductivity of Cagniard. It can be proved quite generally (Weidelt, 1972) that the real part of $Z_1(0)$ is the mean depth of the in-phase current system flowing in a horizontally stratified conductor, and Schmucker has argued that according to eq. 112 the depth $d_1 + \frac{1}{2}\delta_2$ at which the current flow is concentrated is in a region whose conductivity can be obtained (through δ_2) from the imaginary part of $Z_1(0)$. Thus by measuring the surface impedance for a number of frequencies and using:

$$h^* = \mathrm{Re}\, Z_1(0)\,, \quad \sigma_c^{-1} = 2\omega\mu_0[\mathrm{Im}\, Z_1(0)]^2 \qquad (113)$$

to determine depth of current flow h^* and conductivity σ_c, respectively, Schmucker has found a remarkably simple method of estimating the earth's conductivity at various depths.

Finally we consider a model in which the upper layer has the greater conductivity. With the chosen conductivity ratio σ_1/σ_2 = 4000, the model represents, in particular, a sea of depth d_1. Fig. 1A and B depict how the amplitudes and phases of the ratios $Z_1(0)$, $S_1(d_1)$ and $T_1(d_1)$ depend on the depth d_1 of the upper layer for different values of ν. The curves are plotted in dimensionless form with all lengths expressed in units of δ_1. Thus they apply for all frequencies and for all conductivities in the given ratio. The impedance curves are similar to those plotted by Price (1962), and reveal quite clearly how the assumption that the source is uniform (ν=0) can fail if the upper layer is shallow enough. This applies not only to the highly non-uniform source for which $\nu\delta_1$ = 1 but also, albeit less markedly and for shallower layers, to a source ($\nu\delta_1$=0.1) which is uniform over a much greater horizontal range than the skin depth of the upper layer. However, if the model represents the sea and the sources are of natural origin, the parameters are such that the assumption ν = 0 is practically always valid (McCann and Price, 1965).

The value $T_1(d_1)$ describes the attenuation of the horizontal magnetic field within the layer by expressing the field at the bottom as a fraction of its surface value, and $S_1(d_1)$ does likewise for the electric and vertical magnetic components. It is very apparent that, whereas the horizontal magnetic field is sharply attenuated even in layers of only a fraction of a skin depth in thickness, the other components are barely changed in shallow layers. When the layer thickness exceeds δ_1, however, the skin effect begins to take over and the attenuation of all components becomes large. The greater the non-uniformity of the source the less pronounced are these features, for the curves for $S_1(d_1)$ and $T_1(d_1)$ tend towards each other as ν increases. Price (1965) has discussed the behavior of electromagnetic fields within seawater by computing values of $S_1(z)$ and $T_1(z)$ for a two-layer earth in which he took $\sigma_2 = 0$. His calculations for different values of z and a particular value of ν illustrate precisely how the field penetrates downwards through the seawater.

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