Mr. Dennison's assumptions of a state of complete rest before the arrival of the wave are the more logical ones. I have confirmed his results, solving the problem through the application of the Laplace transform. For completeness, I give here the expression for the response of a receptor to a sine-wave displacement input of amplitude $A$ and frequency $n/2\pi$:

$$\theta(t) = nA[(N^2 - n^2)^2 + K^2n^2]^{-1/2}$$

$$\cdot \left\{ n \sin \left[ nl - \arctan \left( \frac{Kn}{N^2 - n^2} \right) \right] \right.$$  

$$+ N^2 (N^2 - K^2)^{-1/2} e^{K(t/2)} \sin \left[ (N^2 - K^2/4)^{1/2} t + \arctan \frac{-K}{2N^2 - K^2} \arctan \frac{K(N^2 - K^2/4)^{1/2}}{N^2 - n^2 - K^2/2} \right]\right.$$  

This answer gives the amplitude, $\theta_0$, and phase, $\phi$, as expressed in Mr. Dennison's discussion.

COMMENTS ON THE DISCUSSION BY A. T. DENNISON

STEPHEN SZASZ*

Prescott's derivation of the differential equation for $\theta$, which describes the movement of the receptor mass with respect to the frame, is correct (equation (6)); the general solution of this equation (7) is also correct. Dennison also agrees with these equations.

Equation (6) being of second order, its general solution contains two arbitrary constants, $\theta_0$ and $\phi$. To find a particular solution applicable to a given physical system, two conditions must be given which, substituted in equation (7), will yield two ordinary equations which determine the numerical values of the arbitrary constants.

Both Prescott and Dennison agree on the first of these initial conditions, namely, $\theta = 0$ for $t = 0$. They disagree, however, on the proper second condition.

Prescott's second condition, $\theta_0 = A \sqrt{C_1^2 + C_2^2}$, states that at $t = 0$, the amplitude of the transient term is equal to the amplitude of the steady-state term. No justification is given in physical terms for this condition.

Dennison's second condition is $-\dot{\theta} = \dot{\alpha}$, which means that at $t = 0$, the velocity of the receptor mass with respect to a system of coordinates fixed in space is zero. This condition is justified on physical grounds.

From here on, both Prescott's and Dennison's calculations are correct. However, because they describe different systems, it is not surprising that the results are different.

Dennison's criticism, therefore, should be directed not at Prescott's mathematics but at the physical validity of the second initial condition in Prescott's paper.

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A DISCUSSION OF THE "FAULT" AND "DIKE" PROBLEMS IN MAGNETOTELLURIC THEORY

J. T. WEAVER*

In two recent papers appearing in Geophysics, d'Erceville and Kunetz (1962) and Rankin (1962) have dealt with the magnetotelluric theory for a plane earth which contains a certain type of vertical fault. In both cases the results depend on a boundary condition which requires the assumption that the normal component of current density vanishes at the surface of the earth. While d'Erceville and Kunetz confine their attention to the region below the surface and thereby avoid explicit mention of the source field, Rankin fol-
discuss Cagniard (1953) by considering a plane-polarized electromagnetic wave normally incident on the surface of the earth. In this case, the assumed boundary condition is not correct, as we shall see later; indeed, it actually leads to a contradiction.

It is the purpose of this note to examine the validity of the boundary condition in some detail, and to point out that if the time variations are considered to be quasi-stationary, then in fact a much stronger assumption is implicit in the form of mathematical model chosen by the above authors.

Throughout this discussion we shall only consider media which have the free space values of permeability and permittivity. In the electromagnetic system of units, this means that the electric displacement is $\mathbf{E}/c^2$ (where $\mathbf{E}$ is the electric field and $c$ is the velocity of light), and that the magnetic induction is identical with the magnetic field $\mathbf{H}$. In addition, all field vectors will be assumed to vary harmonically in time with angular frequency $\omega$, so that all time derivatives may be replaced by $i\omega$ if a factor exp $i\omega t$ is understood throughout. With these simplifications, the field vectors inside a continuous medium of conductivity $\sigma$ satisfy Maxwell's equations in the form

$$\text{curl } \mathbf{H} = \omega^2(\gamma^2 + i\kappa^2)\mathbf{E}, \quad (1)$$

$$\text{curl } \mathbf{E} = -i\omega\mathbf{H}, \quad (2)$$

where $\gamma^2 = 4\pi\rho\omega$ and $\kappa = \omega/c$. If the field variations are sufficiently slow so that $\kappa^2/\gamma^2 \ll 1$, an approximate form of (1) is

$$\text{curl } \mathbf{H} = (\gamma^2/\omega)\mathbf{E}, \quad (3)$$

which, together with (2) and div $\mathbf{H} = 0$, yields the diffusion equation

$$(\nabla^2 - i\gamma^2)\mathbf{H} = 0. \quad (4)$$

The above approximation fails, even for slowly varying fields, if the medium has negligible conductivity. In this case (1) becomes

$$\text{curl } \mathbf{H} = i(k^2/\omega)\mathbf{E}, \quad (5)$$

from which we obtain the wave equation

$$(\nabla^2 + k^2)\mathbf{H} = 0. \quad (6)$$

d'Erceville and Kunetz take the plane $z = 0$ as the surface of the earth, with the $z$-axis directed downwards, and the plane $z = 0$ representing a vertical fault, dividing the earth into two regions of conductivity $\sigma_1(x < 0)$ and $\sigma_2(x > 0)$ respectively. Since the region $z < 0$ is occupied by the nonconducting atmosphere, the field vectors there are solutions of the wave equation (6), while inside the earth ($z > 0$) they satisfy the diffusion equation (4), where the constant $\gamma$ has a different value in the two regions of different conductivity. Rankin's model is the same except that in place of a single fault, he considers a dike, of conductivity $\sigma_d$, situated between the planes $x = \pm l/2$, the conductivity on either side of the dike being $\sigma_2$. Both models also include a common underlying medium of conductivity $\sigma$, but we shall disregard it in this discussion since its presence has no effect on the arguments to be presented here.

In addition, two simplifying assumptions are made by these authors. They are (1) all quantities are independent of the variable $y$, and (2) the magnetic vector is everywhere in the $y$-direction. Under these conditions, and with $i$, the current density, introduced through the relation $i = \sigma \mathbf{E}$, the $z$-component of equation (3) becomes

$$\partial H_y/\partial x = 4\pi i.$$

If it is now assumed that $i_x = 0$, it follows immediately that $H_y$ is constant on $z = 0$, despite the change of conductivity at a discontinuity.

However, this conclusion contradicts the results obtained by considering a plane wave normally incident on $z = 0$. To show this, we consider first the plane-polarized wave

$$H_y^r = Ae^{-i\omega t}, \quad E_z^r = -B e^{-i\omega t},$$

and a transmitted field which, being a solution of (4), can be written in the form

$$H_y^t = Ce^{-i\omega t}, \quad E_z^t = (\omega\sqrt{i}/\gamma)Ce^{-i\omega t}.$$

Fulfillment of the boundary conditions specifying the continuity of $H_y$ and $E_z$ at $z = 0$ requires

$$B = i(1 - k\sqrt{i}/\gamma)/1 + k\sqrt{i}/\gamma = i(1 - 2k\sqrt{i}/\gamma) + O(k^2/\gamma^2)$$

and

$$C = 2A/(1 + k\sqrt{i}/\gamma) = 2A(1 - k\sqrt{i}/\gamma) + O(k^2/\gamma^2).$$

Since the terms $O(k^2/\gamma^2)$ are negligible compared with unity, the magnetic field at the surface of the earth is given by

$$H_y(0, z) = 2A(1 - k\sqrt{i}/\gamma). \quad (7)$$

Now in considering the vertical fault model, d'Erceville and Kunetz assume that the solutions for the magnetic fields in their respective regions within the earth can be written in the form

$$H_z(x, z) = P_i(x, z), \quad (8)$$

where $H_z$ is the solution for a uniform earth of conductivity $\sigma$, and $P_i$ is a perturbation term tending to zero with increasing distance from the fault. (The subscript, $i$, takes the value 1 or 2 depending on the region referred to.) If it is now assumed that these fields are due to a plane wave normally incident on the earth, it follows from (7) that the total magnetic field on $z = 0$ is

$$2A(1 - k\sqrt{i}/\gamma_i) + P_i(x, 0). \quad (9)$$
Since \( P_t \to 0 \) as \( x \to \pm \infty \), we have from (9),

\[
\left[ H_y \right]_{z=0} \to 2A \left( 1 - k \sqrt{i/\gamma} \right) \quad \text{as} \quad x \to -\infty
\]

and

\[
\left[ H_y \right]_{z=0} \to 2A \left( 1 - k \sqrt{i/\gamma} \right) \quad \text{as} \quad x \to +\infty,
\]

and since \( \gamma_1 \neq \gamma_2 \), these results are clearly incompatible with the requirement that \( H_y \) be constant on \( z=0 \).

Actually, the solutions of d'Erceville and Kunetz are derived without any consideration of the field external to the earth, and are therefore merely dependent on the earth. Now if one is considering electromagnetic wave propagation there. Actually, however, it can be seen that (for quasi-stationary fields) \( H_y \) is constant throughout the whole region \( z \leq 0 \), and not only at the surface \( z=0 \). Thus, under those conditions when the normal component of current density at the surface of the earth can be regarded as negligible, it is actually implicit in the form of mathematical model chosen that the total (inducing plus induced) magnetic field is constant everywhere above the earth. This is a much stronger assumption than is immediately apparent from reading the papers of d'Erceville and Kunetz and of Rankin, but it is a necessary one if their results are to remain valid.

We conclude, therefore, that only if the magnetic field above the earth is quasi-stationary and uniform are the solutions of the above authors correct. Further to this, it must be pointed out that the uniformity requirement is a very restrictive one, which considerably limits the usefulness of the results in applications. In fact, the theory of Cagniard (1953) was criticized on this very point by Wait (1954). He showed that the results based on a uniform field assumption would only be applicable to situations in which the source field originated with vast ionospheric current sheets whose horizontal dimensions were larger than the "skin depth" in the ground. More recently, Price (1962) has indicated that if the conductivity is taken to vary with depth (as is done, in fact, in the models of d'Erceville and Kunetz, and Rankin), then a uniform field assumption can yield very inaccurate results even if the ionospheric currents are of global dimensions. It seems, therefore, that some caution must be exercised in trying to apply the theoretical solutions for the "fault" and "dike" models to a practical situation.
**REFERENCES**


**REPLY OF MESSRS I. D'ERCEVILLE* AND G. KUNETZ* TO THE DISCUSSION BY J. T. WEAVER**

The authors have read Mr. Weaver's remarks with great interest. They would like to point out that their aim has not been to clear up the still debated question of how to define the primary field that generates the telluric currents, but to study the effect of a fault on a special type of field, complying with the laws of electromagnetism (neglecting the displacement currents, as is usual at the frequencies considered) and differing little from actual telluric currents.

Moreover, the results show that the effect of a fault has a rather limited extension, so that in this case “the infinite is quite near.” To get valid results, it will then be sufficient that the telluric current complies with the type considered in an area near the observation point.

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**REPLY BY D. RANKIN* TO THE DISCUSSION BY J. T. WEAVER**

I am indebted to Weaver if he has indeed clarified certain points which I had previously considered to be obvious. Cagniard (1953) states explicitly the magnitude of the wavelengths in free space and it is further implicit in the work of Rankin (1962) that it is indeed this same electromagnetic field which is being considered. The plane wave aspect of the problem arises from the extent of and not the distance from the source so that truly it is the induction field and not the radiation field that is under discussion. I had believed, until this note by Weaver, that d'Erceville and Kunetz (1962) also considered a plane wave incident on the earth and in fact that I was merely following both Cagniard and d'Erceville and Kunetz in this matter. The consistency of the results would tend to confirm this belief.

The last two formulae of the appendix in the work of Rankin (1962) give explicitly the approximation which is implicit throughout the paper and I believe also in d'Erceville and Kunetz's work. Comparison with Weaver's formulae:

\[
\left| I_y \right|_{x=0} \to 2 \hat{A} \left( 1 - k \sqrt{\tau_1} \right) \quad \text{as} \quad x \to + \infty
\]

and

\[
\left| I_y \right|_{r=0} \to 2 \hat{A} \left( 1 - k \sqrt{\tau_1} \right) \quad \text{as} \quad r \to + \infty
\]

shows that

\[
\left| I_y \right|_{r=0} = \left| I_y \right|_{r=\infty}
\]

In this same approximation \( \xi = 0 \) and the consequence that \( H_y \) is constant across the trace of a fault or dike thus follows even as in the reformulation by Weaver. An experimental verification of this result is contained in a forthcoming paper by the author.

**REFERENCES**


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