I. Notes on the Electromagnetic Induction within the Earth.

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(Read Feb. 19 and June 18, 1946.—Received June 30, 1946.)

1. Introduction

It is well known that the changes in earth current closely correlate with those of the earth's magnetic field. Many geophysicists hitherto endeavoured to explain this fact. Nowadays, it is established that the greatest part of the earth current is the induced electricity within the earth by the variation of the geomagnetic force of external origin.

Chapman and Whitehead\(^1\) investigated the electromagnetic induction by the diurnal variation of the external magnetic field. They considered a model of the earth containing a uniformly conducting core surrounded by a non-conducting layer. According to them, the diurnal variation of earth current at Ebro\(^3\), Spain, derived from the magnetic potential function, agreed roughly with the observation, while the amplitude was several times smaller.

Erte\(^3\), considering the heterogeneity of the electrical conductivity, extended Chapman–Whitehead's theory. The same problem in the anisotropic earth were studied by Hirayama\(^5\) as a two-dimensional problem.

In this paper, a method to discuss the relation between the variation of geomagnetism and of earth current is described. To avoid troublesome mathematics, the writer simply treats the earth as semi-infinite media. (This assumption will be reasonable when the discussion is limited to a narrow region.)

We can obtain the intensity of magnetic or electric field as the solution of Maxwell's equation in quasi-stationary state. Taking into consideration the initial condition and the boundary condition at the surface of the earth where the magnetic field or earth current are observed, we can get the variation of the magnetic field from the observed variation of earth current or vice versa.

By this method, the writer discussed the diurnal variation of earth current at Ebro as Chapman and Whitehead did.

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At first, assuming that the electrical conductivity of the earth is isotropic, the diurnal variation of earth current is calculated from the observed diurnal variation of geomagnetic field. Then, comparing the observed value to the calculated ones, the electrical conductivity of the earth is determined.

The writer further treats the same problem in the anisotropic earth. Analysing the data of earth current, in this case, the variation of geomagnetic field is calculated.

In order to avoid the accidental irregularity of the observation, the hourly values of the diurnal variation averaged through the international calm days of the year 1927 are used.

The variations are shown in Fig. 1.

2. Electromagnetic induction within semi-infinite, isotropic earth.

Electric and magnetic quantities in earth are connected by Maxwell's equations

\[ \text{rot } \vec{H} = 4\pi \vec{i}, \quad (1) \]
\[ \text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2) \]

for the phenomena are quasi-stationary, where \( \vec{H}, \vec{B}, \vec{E} \) and \( \vec{i} \) denote magnetic field, magnetic induction, electric field and electric current-density respectively. Between these quantities we have the next relations,

\[ \vec{i} = \sigma \vec{E}, \quad (3) \]
\[ \vec{B} = \mu \vec{H}, \quad (4) \]

and besides

\[ \text{div } \vec{B} = 0 \quad (5) \]

where \( \sigma \) and \( \mu \) are specific electrical conductivity and magnetic permeability of earth respectively. In this section, it is assumed that the earth is isotropic and homogeneous, so these quantities are both scalar and constant everywhere.

Eliminating \( \vec{E} \) from these equations, we have

\[ \frac{\partial \vec{H}}{\partial t} = \kappa \vec{p} \cdot \vec{H} \quad (6) \]
where

\[ \kappa^2 = 1/4 \pi \sigma \mu. \]  

(7)

In our problem, as the constant part of the earth's magnetic field does not contribute to the variation of electric field in the earth, we can take the next relation as the initial condition without loss of generality,

at \( t=0 \), \( H = 0 \)

(8)

and the boundary condition is

at the surface (\( z=0 \)), \( H = H_0(t) \)

(9)

while the axes \( x, y \) and \( z \) are taken to be north-, east-, and downwards respectively.

Then the problem becomes equivalent to the typical one in one-dimensional heat conduction in a semi-infinite solid whose solution can be expressed in the well-known form

\[ H = \frac{2}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\beta^2} H_0 \left( t - \frac{\beta^2}{4 \kappa^2 \beta^2} \right) d\beta. \]

When the magnetic field at the surface can be expressed by such a series of harmonic functions as

\[ H_0 = \sum_{n=1}^{\infty} a_n \sin(n \pi t), \]

(11)

the magnetic field in the earth becomes

\[ H = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2}{4 \kappa^2 \beta^2}} \sin \left( \frac{\pi z}{2 \kappa} \sqrt{\frac{2p}{s^2}} \right) \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2}{4 \kappa^2 \beta^2}} \sin \left( \frac{\pi z}{2 \kappa} \sqrt{\frac{2p}{s^2}} \right) d\beta. \]

(12)

Then, considering the equation (2), we can get the intensity of the electric field from the above equation. If we assume that \( H \) does not depend both on \( x \) an \( y \), the vertical component of the electric field vanishes everywhere, while the north and east components at the surface become

\[ E_{n} = \sqrt{\frac{\mu_0}{4 \pi \sigma}} \sum_{n=1}^{\infty} a_n \sin \left( \frac{\pi z}{2 \kappa} \sqrt{\frac{2p}{s^2}} \right), \]

\[ E_{e} = \sqrt{\frac{\mu_0}{4 \pi \sigma}} \sum_{n=1}^{\infty} a_n \sin \left( \frac{\pi z}{2 \kappa} \sqrt{\frac{2p}{s^2}} \right). \]

(13)

When the hourly values of the diurnal variation of the north and east components of the electric field are required, the results of the above equation can be used.
components of the earth's magnetic field are given, we can obtain the numerical values of the coefficients \(a_x, \ldots, a_y, \ldots\) by means of harmonic analysis. Then, combining these values, we can calculate the series of the right-hand side of (13) and so, considering the hourly values of the earth potential, the most probable values of \(\sqrt{\frac{\mu D}{4\pi \sigma}}\) will be determined by means of least squares. Since \(p = \frac{\pi}{24\times60\times60}\) and practically \(\mu = 1\), then, we can determine the most probable values of \(\sigma\).

In this article, as it is assumed that the earth is electrically isotropic, specific values of the electrical conductivity can be determined independently from either combinations of the east component of geomagnetic force and the north component of earth current or the ones of north component of geomagnetic force and the east component of earth current, denoting the former \(\sigma_x\) and the latter \(\sigma_y\) respectively.

At Ebro, the amplitude of the east component is about three times larger than the north component in geomagnetic diurnal variation and vice versa in earth current. The determined value \(\sigma_y\), therefore, will be more reliable than \(\sigma_x\) for the disturbance in earth current, which will be caused by some other origins such as contact electromotive forces at the surfaces of the electrodes, will affect equally to the diurnal variation of earth current. In 1927, the determined value of \(\sigma_y\) and \(\sigma_x\), expressed in terms of \(10^{-6}\) ohm\(^{-1}\) cm\(^{-1}\), is \(4.7\pm0.9\) and \(3.3\pm1.8\), where the probable errors are comparatively large because the errors in \(\sqrt{\frac{\mu D}{4\pi \sigma}}\) become twice in \(\sigma\).

In the next place, we can obtain the diurnal variation of earth current using the above calculated values of the electrical conductivity of the earth, as shown in Fig. 2. The calculated figures roughly agree with the observation.

About twenty years ago, Chapman and Whitehead\(^6\) computed theoretically the diurnal variation of earth current at Ebro. They determined several coefficients of

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6) Chapman and Whitehead, _loc. cit._
s herical harmonic expansion of geomagnetic potential by analysing the geomagnetic data (averaged diurnal variation from 1914 to 1918) observed at several stations distributed all over the world, and then computed the diurnal variation of earth current on the stand-point of electromagnetic induction theory. The diurnal variations obtained by their investigations are shown in Fig. 3. The figures well resemble with those in Fig. 2. As the annual means of geomagnetic forces do not extraordinarily differ every year, it is reasonable that we obtain similar conclusions with their investigation, when the earth can be treated as semi-infinite media. In Fig. 2, there are somewhat systematic differences between the observed figures and the computed ones as well as in Fig. 3.

If we eliminate $H$ in Maxwell's equation, the process of the above mentioned theory will be also available for the discussion of the variation of the magnetic field from the observed variation of earth current.


It has been often reported$^{7,8,9}$ that the variation of earth current does not occur in the direction perpendicular to that of the earth's magnetic field. Consequently, there must exist apparent anisotropy of the electrical conductivity of earth. This anisotropy is observed even by direct measurements$^{10,11}$. Now we shall extend the electromagnetic induction theory to the anisotropic earth.

In this case, Ohm's law becomes

where ( ) means tensor.

Eliminating $\vec{H}$ from Maxwell's equation, we get

$$4\pi(\sigma+\mu)\frac{\partial \vec{E}}{\partial t} = p + \text{grad div } \vec{E}$$

(15)

If we assume that one of the directions of the principal axes coincides with the vertical line to the surface, each component of electric field must satisfy the next equations

$$
\begin{align*}
\frac{\partial E_x}{\partial t} &= \kappa_x \frac{\partial^2 E_x}{\partial z^2} \\
\frac{\partial E_y}{\partial t} &= \kappa_y \frac{\partial^2 E_y}{\partial z^2} \\
\frac{\partial E_z}{\partial t} &= 0
\end{align*}
$$

(16)

where $x$, $y$, and $z$ are the directions of the principal axes.

The initial and boundary conditions are

$$\begin{align*}
\text{at } t=0 & \quad \vec{E} = 0 \\
\text{at the surface } & \quad \vec{E} = \vec{E}_0(t)
\end{align*}$$

(17)

as well as in the former article.

When the electric field at the surface can be expressed by a series of harmonic functions such as

$$\vec{E}_0 = \sum_{s=1}^{n-1} A_s \sin s\pi t,$$

(18)

the electric field in the earth becomes

$$
\begin{align*}
E_x &= \sum_{s=1}^{n-1} A_{x_s} e^{-\frac{\pi}{\kappa_x} \sqrt{s^2 + \frac{sp^2}{2}}} \sin \left( s\pi t - \frac{2}{\kappa_x} \sqrt{s^2 + \frac{sp^2}{2}} \right) \\
E_y &= \sum_{s=1}^{n-1} A_{y_s} e^{-\frac{\pi}{\kappa_y} \sqrt{s^2 + \frac{sp^2}{2}}} \sin \left( s\pi t - \frac{2}{\kappa_y} \sqrt{s^2 + \frac{sp^2}{2}} \right)
\end{align*}
$$

Substituting these relations into $\text{rot } \vec{E} = -\mu \frac{\partial \vec{E}}{\partial t}$ and integrating with respect to time, we get the magnetic field. At the surface, the obtained magnetic field is written as follows:

$$
\begin{align*}
H_{x_0} &= \frac{1}{\mu \kappa_x} \sum_{s=1}^{n-1} A_{x_s} \cos \left( s\pi t + \frac{\pi}{4} \right) \\
H_{y_0} &= -\frac{1}{\mu \kappa_y} \sum_{s=1}^{n-1} A_{y_s} \cos \left( s\pi t + \frac{\pi}{4} \right)
\end{align*}
$$

(20)

$\theta$ is the angle between $x$- and $z$- (north)
direction is \( \sigma \), as appears in Fig. 4. we can write the north and east components of the magnetic field at the surface in the next expressions:

\[
H_{0x} = \frac{1}{\mu} \left( \frac{1}{\kappa_y} - \frac{1}{\kappa_x} \right) \frac{\sin 2\alpha}{2} \frac{A_{0x}}{\sqrt{sp}} \cos \left( sp + \frac{\pi}{4} \right) \\
+ \left( \frac{1}{2} \left( \frac{1}{\kappa_y} + \frac{1}{\kappa_x} \right) + \frac{\cos 2\alpha}{2} \left( \frac{1}{\kappa_y} - \frac{1}{\kappa_x} \right) \right) \sum_{s=1}^{n-1} \frac{A_{sx}}{\sqrt{sp}} \cos \left( sp + \frac{\pi}{4} \right) 
\]

\[
H_{0y} = -\frac{1}{\mu} \left( \frac{1}{2} \left( \frac{1}{\kappa_y} + \frac{1}{\kappa_x} \right) - \frac{\cos 2\alpha}{2} \left( \frac{1}{\kappa_y} - \frac{1}{\kappa_x} \right) \right) \sum_{s=1}^{n-1} \frac{A_{sy}}{\sqrt{sp}} \cos \left( sp + \frac{\pi}{4} \right) \\
- \frac{1}{\mu} \left( \frac{1}{\kappa_y} - \frac{1}{\kappa_x} \right) \frac{\sin 2\alpha}{2} \sum_{s=1}^{n-1} \frac{A_{sx}}{\sqrt{sp}} \cos \left( sp + \frac{\pi}{4} \right)
\]

(21)

where

\[
\sum_{s=1}^{n-1} A_{sx} \sin sp = E_{0x}, \quad \sum_{s=1}^{n-1} A_{sy} \sin sp = E_{0y}.
\]

(22)

Then, in the same manner with those of the isotropic case, we can determine \( \sigma, \kappa_z \) and \( \kappa_y \) or consequently \( \sigma_{xx} \) and \( \sigma_{yy} \).

As a result of the actual determination, we get from the data of 1927:

- Direction of minimum \( \sigma \)
  - \( \sigma_{\min} = 0.32 \times 10^{-6} \text{ ohm}^{-1} \text{ cm}^{-1} \),
  - \( \sigma_{\max} = 2.0 \times 10^{-6} \text{ ohm}^{-1} \text{ cm}^{-1} \),
  - \( \sigma_{\min}/\sigma_{\max} = 0.16 \).

Using these values, the diurnal variation of the earth's magnetic field expected from the theory is shown in Fig. 5. The observation agrees roughly with the calculation.

In the present case, the procedure of the determination of the electrical conductivity is much more complicated than in the isotropic case. For this reason, the obtained values will be less reliable. In both cases, the determined values of the electrical conductivity agree in their order.

As appear in Fig. 6, the direction of variation of earth current is nearly N 20° W, while that of magnetic field is E-W. When the earth is isotropic, the induced currents flow to the direction perpendicular to the geomagnetic variation. Therefore, the present deviation of the direction of earth current

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**Fig. 5**

Upper Figures—Diurnal variation of \( X \), 1927.
Lower Figures—Diurnal variation of \( Y \), 1927.
(Full and broken lines correspond respectively to the observation and the calculation)
will be due to such anisotropy of the electrical conductivity as the direction of minimum \( \sigma = N18^\circ W \) as was given by the theory mentioned above. It is reasonable that the earth potential predominates in the direction of high resistance. The determination will become more accurate if we use magnetic variations (such as "bay-ty. e."") disturbances) whose directions would be nearly parallel to that of minimum \( \sigma \).

4. Conclusion.

A practical method to discuss the relation between the variation of the earth's magnetic field and that of earth current was described in this paper both in isotropic and anisotropic earth.

Although the writer assumes that the earth is semi-infinite media, the earth current calculated from the observed variation of geomagnetic field agree fairly well with the one derived from the spherical harmonic analysis of the magnetic potential investigated by Chapman and Whitehead who used the world-wide data.

The anisotropy of the electrical conductivity agrees with the fact that the direction of the diurnal variation of earth current at Ebro is nearly N\(20^\circ\) W because the calculated direction of minimum \( \sigma \) is N\(18^\circ\) W.

Unfortunately, the determination of the electrical conductivity is rather rough for our method is not a direct measurement. For this reason, it seems meaningless to discuss the time-change of the electrical conductivity and its anisotropy by this method. It is desirable to measure these quantities directly and detect their change during a comparatively long period.

Anyhow, the present study shows that there exist some anisotropy in the electrical property of earth as was often reported. About the cause of this anisotropy we have not yet precise knowledge. In strained state earth may show some anisotropy, and besides certain herogenity near the point where the observations are executed may also be a probable cause of the anisotropy.

It will be of interest and importance to investigate the relation between the anisotropy and the mechanical structure of the earth's crust.

The writer wishes to express his sincere thanks to Prof. F. Kishinouye for his advices in the course of this study.
1. 地磁気変化に伴う大地中の電磁感応

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電流の大部分が地球外部に原因を有する地球磁場の変化に伴って誘導されるものであることは多くの地球物理学者に仗て確められている。

本文論においては地表に於て観測された地磁気鉄は地電位差変化の値から、地電位差の地磁気の変化を理論的に求める一方法を提出し、質問と計算との比較から地電位の電磁感応を求め得ることを示した。

はじめに大地は電気的に等方であると仮定した。この場合は互に直角な地磁気及び地電位差の変化を比較することにより、地電位の電磁感応を求めることができる。例として Spain の Ebro の日変化の一年間の平均値を基に計算に用い、地磁気変化より求めた地電位変化は、Cahpman と Whitehead が地磁気の積分の問題を展開から求めたものと形に於て非常によく一致した。

次に最近問題になっている地磁気の異方性を考慮に入れた場合に考慮した。この場合は Maxwell の方程式を解く方法上、地電位の変化をもととして如何なる地磁気の変化が期待されるかを調べた。上述の Ebro の日変化を例として解析を行った結果は、電磁感応最小の方向は N18°W 又最大値と最小値の比は 0.2 程度であった。このことは Ebro の地磁気の変化が殆ど N20°W の方向に限られていることをよく説明するものと云えよう。この場合地磁気の変化は水平でクルト層に於て磁々東西であるから、異方性を求める上にはあまり適していない。例えば地電位変化の方向とは平行に近いような地磁気の変化を用いて求めたならば、更に精度よく異方性を求めることが出来よう。

いずれにしても此方法は間接的であって様々な誤差を含み易いし、精度も不充分であるから、地磁気の性質の時間的変化を調べるには不適であって、直接測定を実施することが望ましい。

E보の例に見る如く、地磁気の性質は少なくとも見掛け上異方性を有する。この異方性を生ずる原因を特に地球物理的な現象特に地震の力学的性質との関係を調べることは、極めて興味あることと考えられ、今後の研究に期待する極大である。