Multitaper Spectral Analysis: State of the Art

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CStat CHARTERED STATISTICIAN
ROYAL STATISTICAL SOCIETY
DATA | EVIDENCE | DECISIONS

Spectral Conflicts

- Desired properties of a spectral estimator
  - Low bias
  - Consistent
  - Efficient (Cramér-Rao lower bound)
  - Statistically characterizable
  - Robust
- NO METHOD CAN SIMULTANEOUSLY ACHIEVE ALL OF THESE GOALS

Two Classes of SpectralEstimator

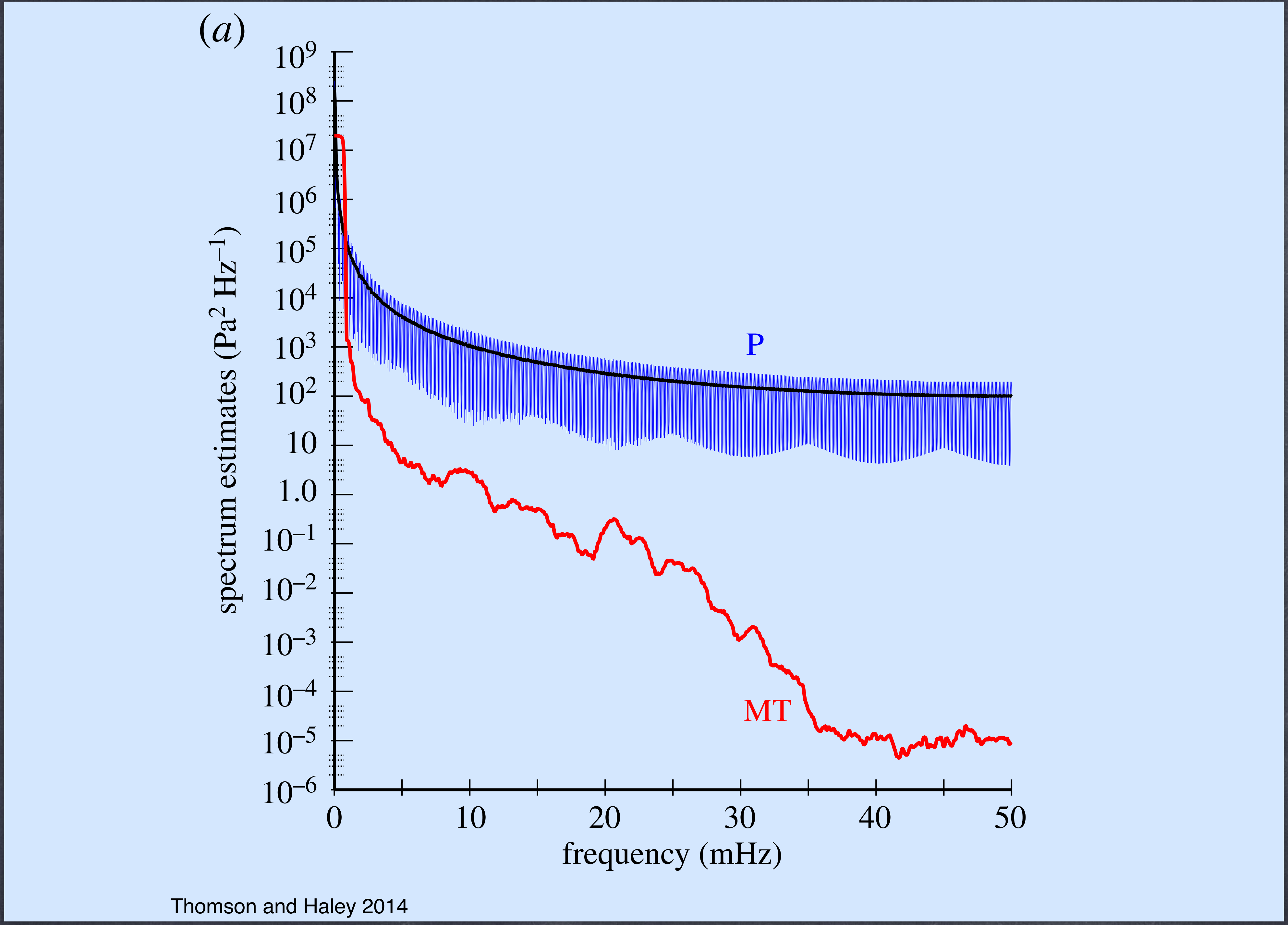
- Parametric
  - Based on a time series model
  - Autoregressive, moving average, ARMA
- Nonparametric
  - Based on Fourier transform
  - Indirect (based on acvs)
  - Direct

Indirect Estimator

- Compute $s_n = \frac{1}{N-n} \sum_{k=0}^{N-1-n} x_k x_{k+n}$
- Multiply by a lag window L(n)
- Take the Fourier transform
- Result is power spectrum by the Wiener-Khintchine Theorem

Indirect Estimator

- Equivalent to extended periodogram
  - Extend N point time series with N+1 zeroes
  - Take Fourier transform and square the coefficients
- $\hat{S}_I(f) = \hat{L}(f) \otimes \left| \frac{\sin(2\pi f)}{2\pi f} \otimes \hat{x}(f) \right|^2$
- Badly biased due to default window
- Obsolete and should never be used



1000 samples of barometric pressure with $\delta t = 10$ s

Blue is periodogram

Red is multitaper estimate with TBW of 8

Direct Estimate

$$\hat{S}_D(f) = \hat{W}(f) \otimes \left| \hat{d}(f) \otimes \hat{x}(f) \right|^2$$

- Uses data taper d (choose your favorite)
- Lag window <-> convolutional smoother W
- Band averaged estimator
- Obsolete and should not be used

WOSA

- Welch overlapped section averaging
- Divide a long time series into short, overlapped sections
- Average Fourier transforms of the sections
- Can be robustified

Summary

- Parametric methods require information that is not usually available
- Indirect methods are obsolete
- Band-averaged direct methods are obsolete
- WOSA is a good choice when frequency of interest is not of order one over the time series length

Multitaper Method

- Proposed by David Thomson in 1982
- Small sample theory with sample size explicit
- Quantifiable bias
- Consistent without ad hoc smoothing
- Resolution is well defined
- High variance efficiency
- Data adaptive
- Line and stochastic components co-exist

Fourier Transform Pair

$$x_n = \int_{-1/2}^{1/2} e^{i2\pi f\left(n-\frac{N-1}{2}\right)} X(f)\,df$$

$$X(f) = \sum_{i=0}^{N-1} x_n e^{-i2\pi f\left(n-\frac{N-1}{2}\right)}$$

X(f) is an entire function of frequency, and not defined only at k/N!

Cramér Representation

Time sequence that is generated by the superposition of random infinitesimal harmonic oscillators has the spectral representation

$$x_n = \int_{-1/2}^{1/2} e^{i2\pi f\left(n-\frac{N-1}{2}\right)} dZ(f)$$

$$\mathcal{E}\left[dZ(f)\right] = \sum_{i=1}^{L} \mu_i \delta(f - f_i)$$

$$\mathcal{E}\left[\left|dZ(f)\right|^2\right] = S(f)df$$

$$\mathcal{E}\left[dZ(f)dZ(f')\right] = 0$$

Fundamental Equation of Spectral Analysis

$$X(f) = \int_{-1/2}^{1/2} \frac{\sin N\pi(f-\nu)}{\sin \pi(f-\nu)} dZ(\nu)$$

Spectral analysis is estimation of the expected value of $|dZ|^2$

Harmonic analysis is estimation of the expected value of dZ

Integral equation of the first kind

First Kind Integral Equation

- Exact solutions do not exist
- Approximate solutions must be sought
- Analogy to inverse problem although problem is quadratic

Generic Integral Equation

$$y(x) = \int_a^b K(x,x')z(x')dx'$$

$$K(x,x') = K(x',x)$$

$$\int_a^b K(x,x')\psi_k(x')\,dx' = \lambda_k \psi_k(x)$$

$$\hat{z}(x) = \sum_k \lambda_k^{-1} \left[\int_a^b y(x')\psi_k(x')dx' \right] \psi_k(x)$$

λ_k->0 as k->N

Slepian Functions

$$\int_{-W}^{W} \frac{\sin N\pi(f-\nu)}{\sin \pi(f-\nu)} U_k(N,W;\nu)\,d\nu = \lambda_k(N,W)\,U_k(N,W;f)$$

$$\int_{-1/2}^{1/2} \frac{\sin N\pi(f-\nu)}{\sin \pi(f-\nu)} U_k(N,W;\nu)\,d\nu = U_k(N,W;f)$$

$$\int_{-W}^{W} U_k(N,W;f)\,U_l(N,W;f)\,df = \lambda_k\,\delta_{kl}$$

$$\int_{-1/2}^{1/2} U_k(N,W;f)\,U_l(N,W;f)\,df = \delta_{kl}$$

Slepian Functions

W or NW is free parameter that defines the inner domain [-W,W)

Eigenvalues are real, distinct and finite in number

$1 > \lambda_0 > ... > \lambda_{N-1}$

First 2NW eigenvalues are nearly 1, then decay exponentially to 0

Eigenvalues give the fractional energy concentration in [-W,W) of the corresponding Slepian function

Slepian Sequences

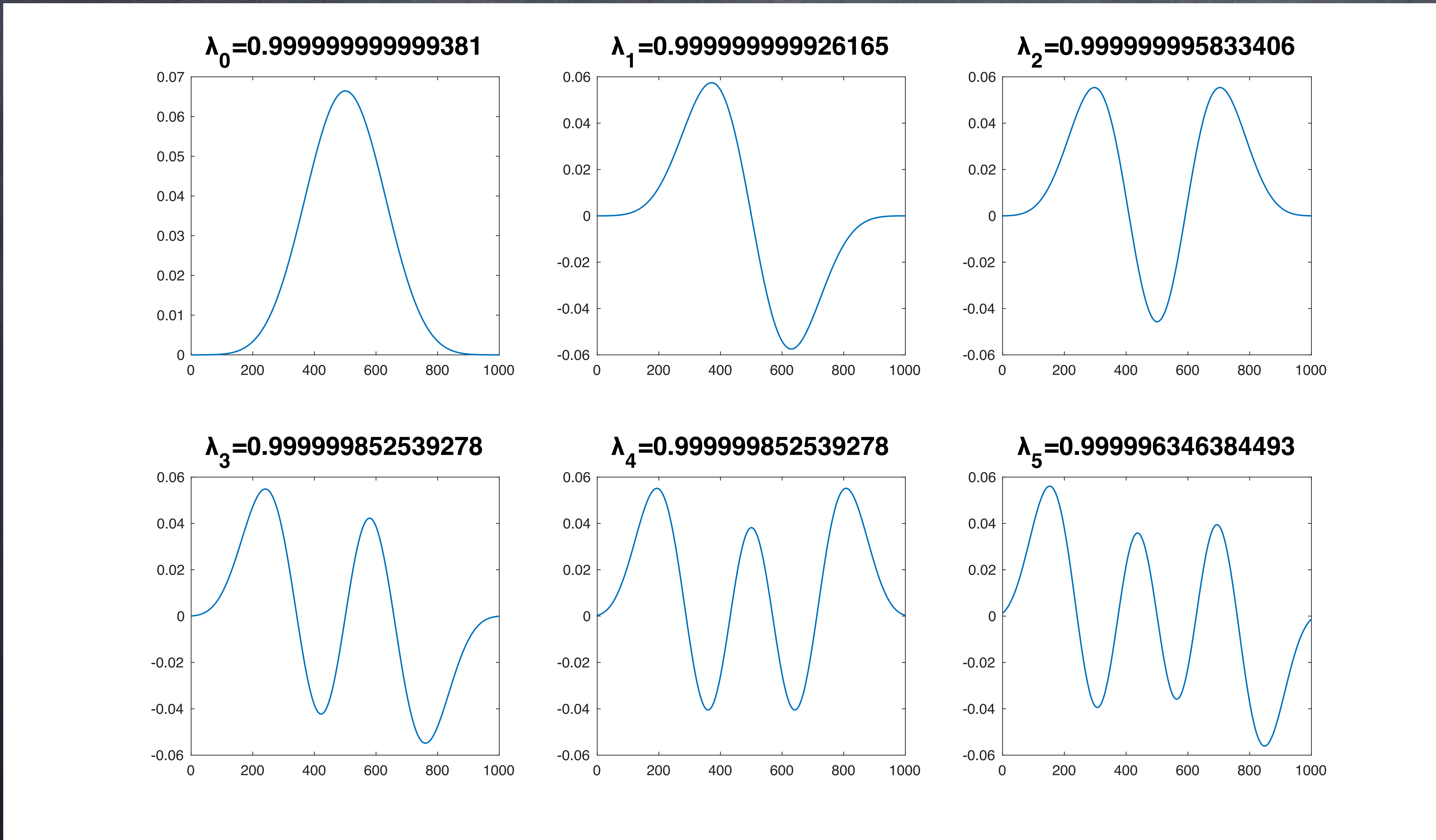
$$v_n^{(k)}(N,W) = \frac{1}{\varepsilon_k \lambda_k(N,W)} \int_{-W}^{W} U_k(N,W;) f\, e^{i2\pi f\left(n-\frac{N-1}{2}\right)} df$$

$$v_n^{(k)}(N,W) = \frac{1}{\varepsilon_k} \int_{-1/2}^{1/2} U_k(N,W;f) e^{i2\pi f\left(n-\frac{N-1}{2}\right)} df$$

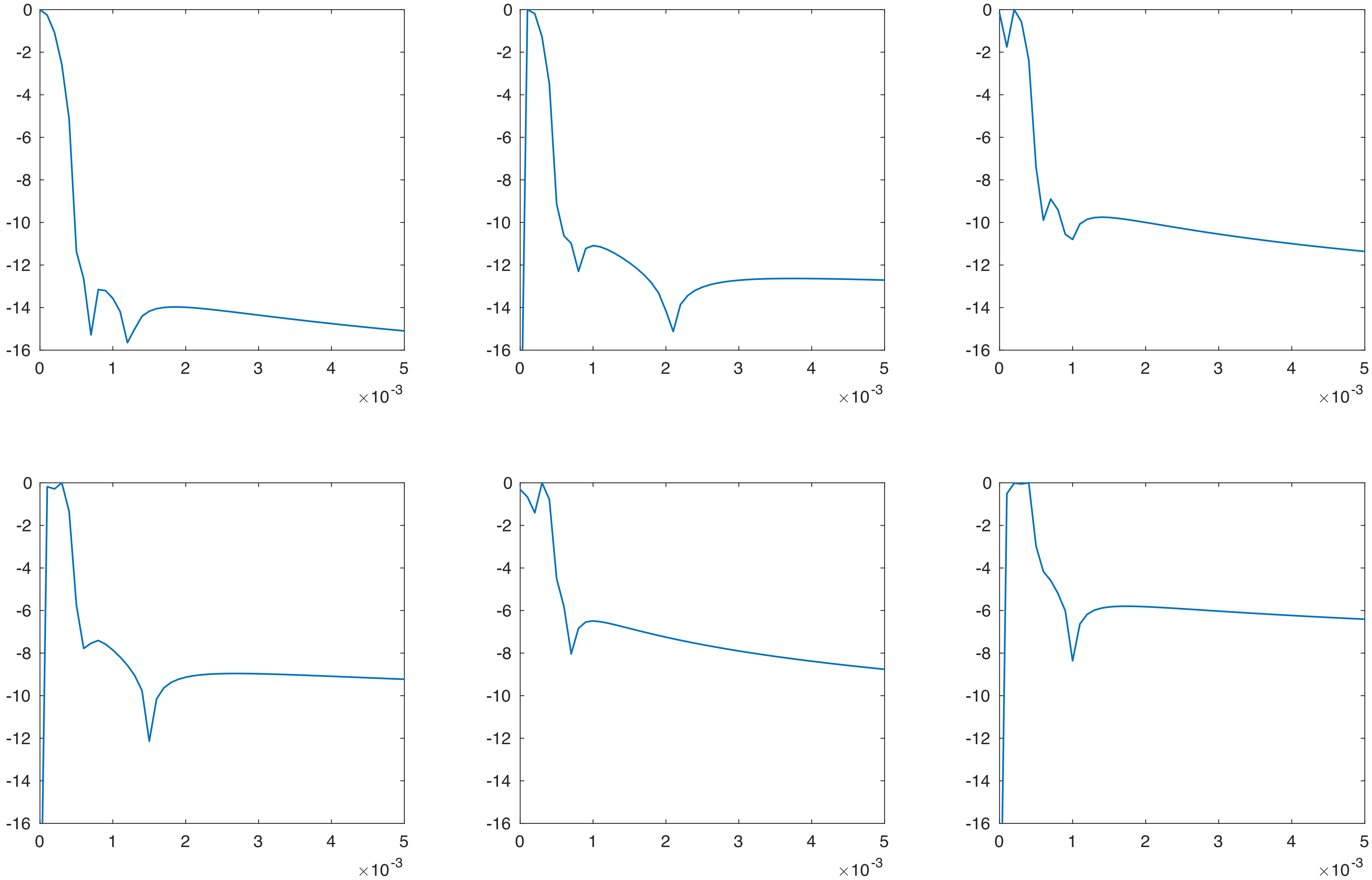
Numerical Solution

- Slepian (1978) gives a tridiagonal analog for the Slepian sequences
- $$U_k(N,W;f) = \varepsilon_k \sum_{n=0}^{N-1} e^{-i2\pi f\left(n-\frac{N-1}{2}\right)} v_n^{(k)}(N,W)$$
- Eigenvalues follow from $\int_{-W}^{W} U_k^2(N,W;f)\,df = \lambda_k$

N=1000 TBW=5



λ0=0.999999999999381
0.07 0.06 0.05 0.04 0.03 0.02 0.01 0
0 200 400 600 800 1000
λ1=0.999999999926165
0.06 0.04 0.02 0 -0.02 -0.04 -0.06
0 200 400 600 800 1000
λ2=0.999999995833406
0.06 0.04 0.02 0 -0.02 -0.04 -0.06
0 200 400 600 800 1000
λ3=0.999999852539278
0.06 0.04 0.02 0 -0.02 -0.04 -0.06
0 200 400 600 800 1000
λ4=0.999999852539278
0.06 0.04 0.02 0 -0.02 -0.04 -0.06
0 200 400 600 800 1000
λ5=0.999996346384493
0.06 0.04 0.02 0 -0.02 -0.04 -0.06
0 200 400 600 800 1000



0 -2 -4 -6 -8 -10 -12 -14 -16
0 1 2 3 4 5
×10-3
0 -2 -4 -6 -8 -10 -12 -14 -16
0 1 2 3 4 5
×10-3
0 -2 -4 -6 -8 -10 -12 -14 -16
0 1 2 3 4 5
×10-3
0 -2 -4 -6 -8 -10 -12 -14 -16
0 1 2 3 4 5
×10-3
0 -2 -4 -6 -8 -10 -12 -14 -16
0 1 2 3 4 5
×10-3
0 -2 -4 -6 -8 -10 -12 -14 -16
0 1 2 3 4 5
×10-3

Multitaper Recipe

- Choose the resolution bandwidth W = r/N
- Fix the upper limit to the number of tapers K ≦ 2NW
- Compute K raw spectra and average their absolute squares frequency-by-frequency

$$a_k\left(f_o\right) = \varepsilon_k \sum_{n=0}^{N-1} v_n^{(k)} x_n e^{-i2\pi f_o\left(n-\frac{N-1}{2}\right)}$$

$$\overline{S}\left(f_o\right) = \frac{1}{2NW} \sum_{k=0}^{K-1} \lambda_k b_k^2 \left|a_k\left(f_o\right)\right|^2$$

Adaptive Weighting

$$\bar{S}(f_o) = \frac{\sum_{k=0}^{K-1} \lambda_k d_k^2(f_o) |a_k(f_o)|^2}{\sum_{k=0}^{K-1} \lambda_k d_k^2(f_o)}$$

$$d_k(f) = \frac{\sqrt{\lambda_k} S(f)}{\lambda_k S(f) + \sigma^2 (1 - \lambda_k)}$$

$$\sigma^2 = \int_{-1/2}^{1/2} S(f) df$$

Prewhitening

- Time domain filter that reduces the spectral dynamic range
- Differentiation is simplest example
- AR filter is better choice
- Useful adjunct

Degrees-of-freedom

$$\nu(f) = 2\sum_{k=0}^{K-1} \lambda_k d_k^2(f)$$

Prewhitening is essential toward maximizing dof

Harmonic Components

$$\hat{\mu}(f) = \frac{\sum_{k=0}^{K-1} U_k(N,W;0)\, a_k(f)}{\sum_{k=0}^{K-1} U_k^2(N,W;0)}$$

Power in line is absolute square with 2 dof

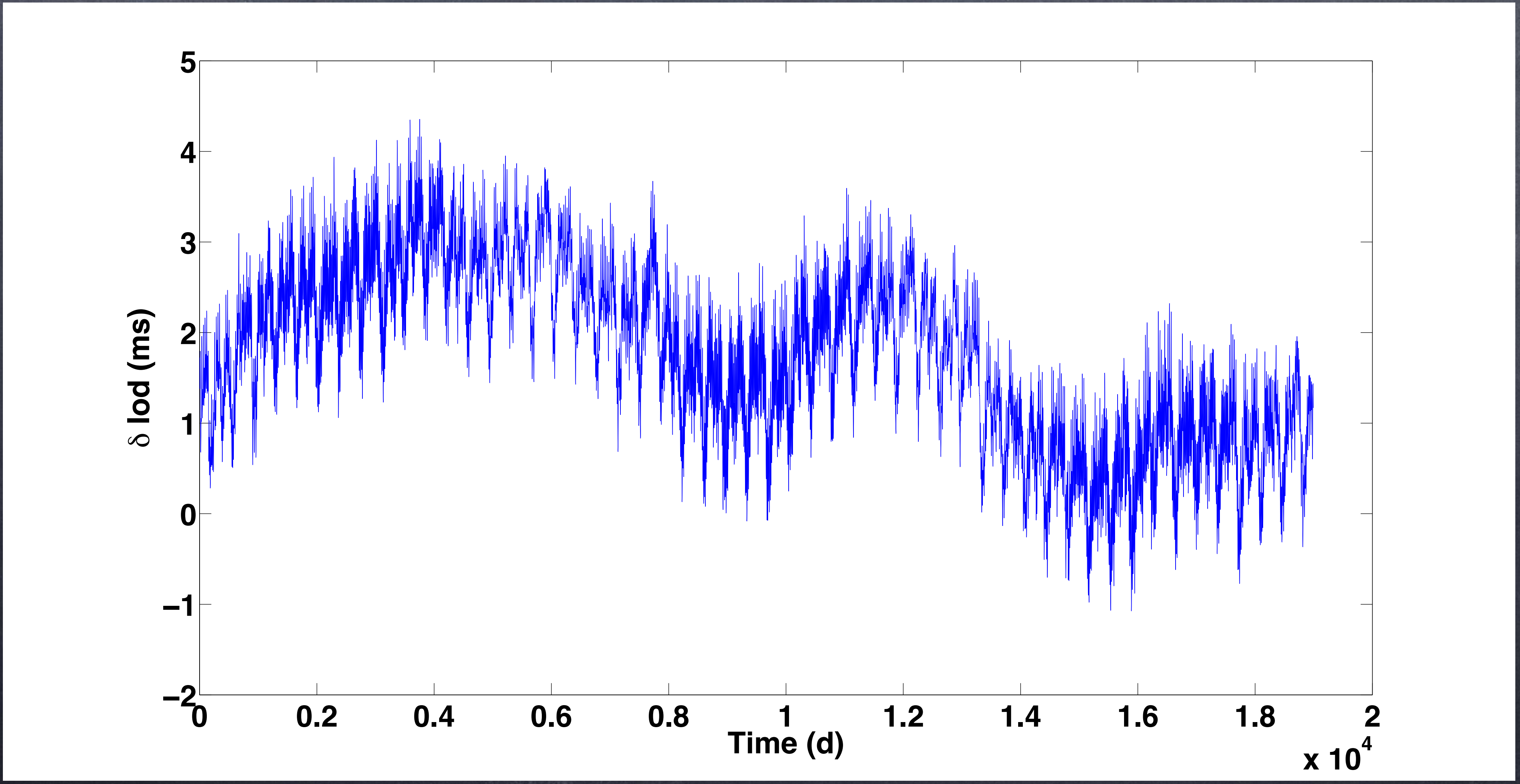
$$\hat{\Sigma}^2(f) = \sum_{k=0}^{K-1} \left| a_k(f) - \hat{\mu}(f)\, U_k(N,W;0) \right|^2$$

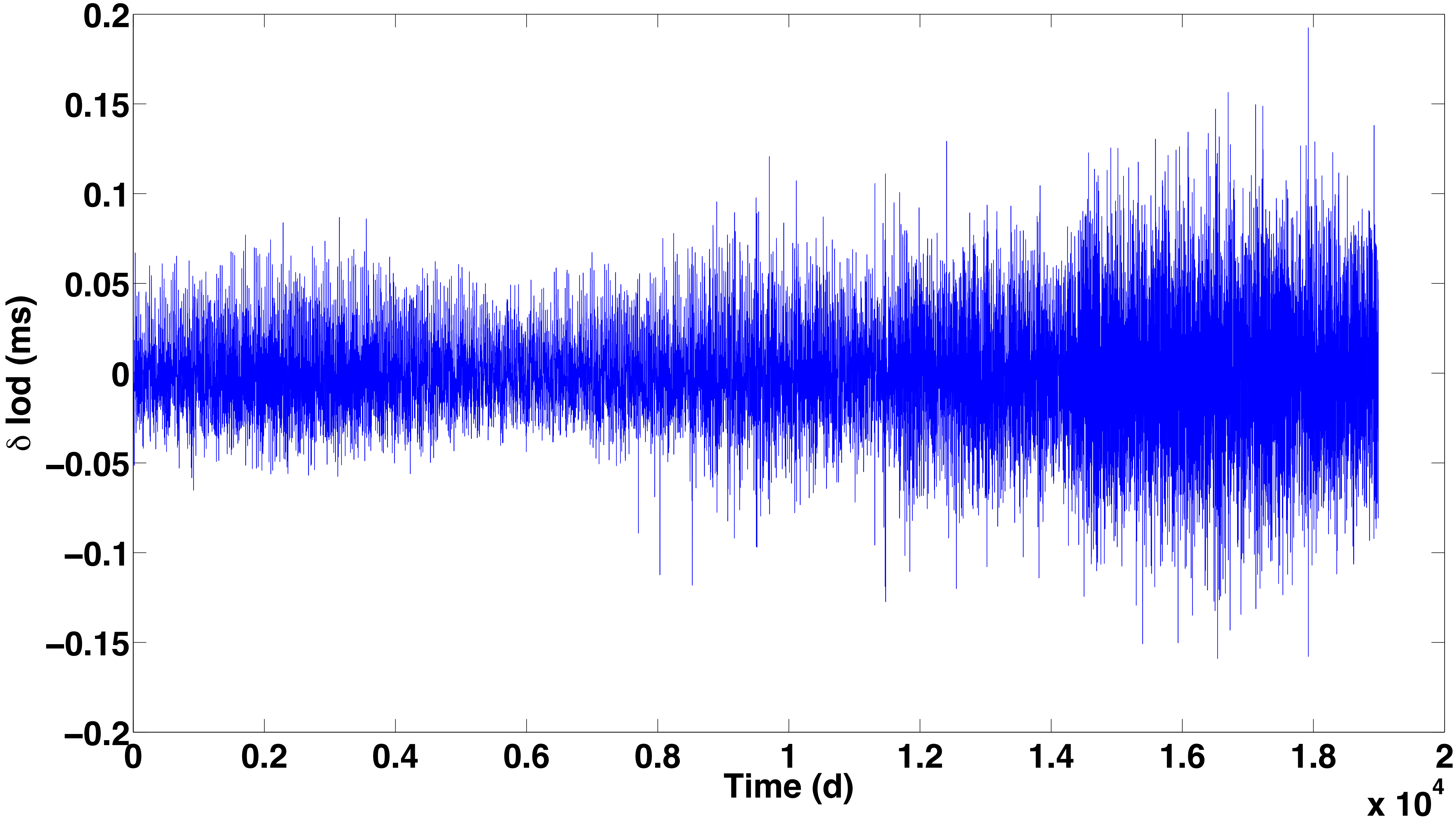
Reshaped spectrum after removing line

$$F(f) = \frac{(\nu(f) - 2) \left| \hat{\mu}(f) \right|^2 \sum_{k=0}^{K-1} U_k^2(N,W;0)}{2 \sum_{k=0}^{K-1} \left| a_k(f) - \hat{\mu}(f)\, U_k(N,W;0) \right|^2} \quad \sim F_{2,\, \nu(f)-2}$$

LOD Data

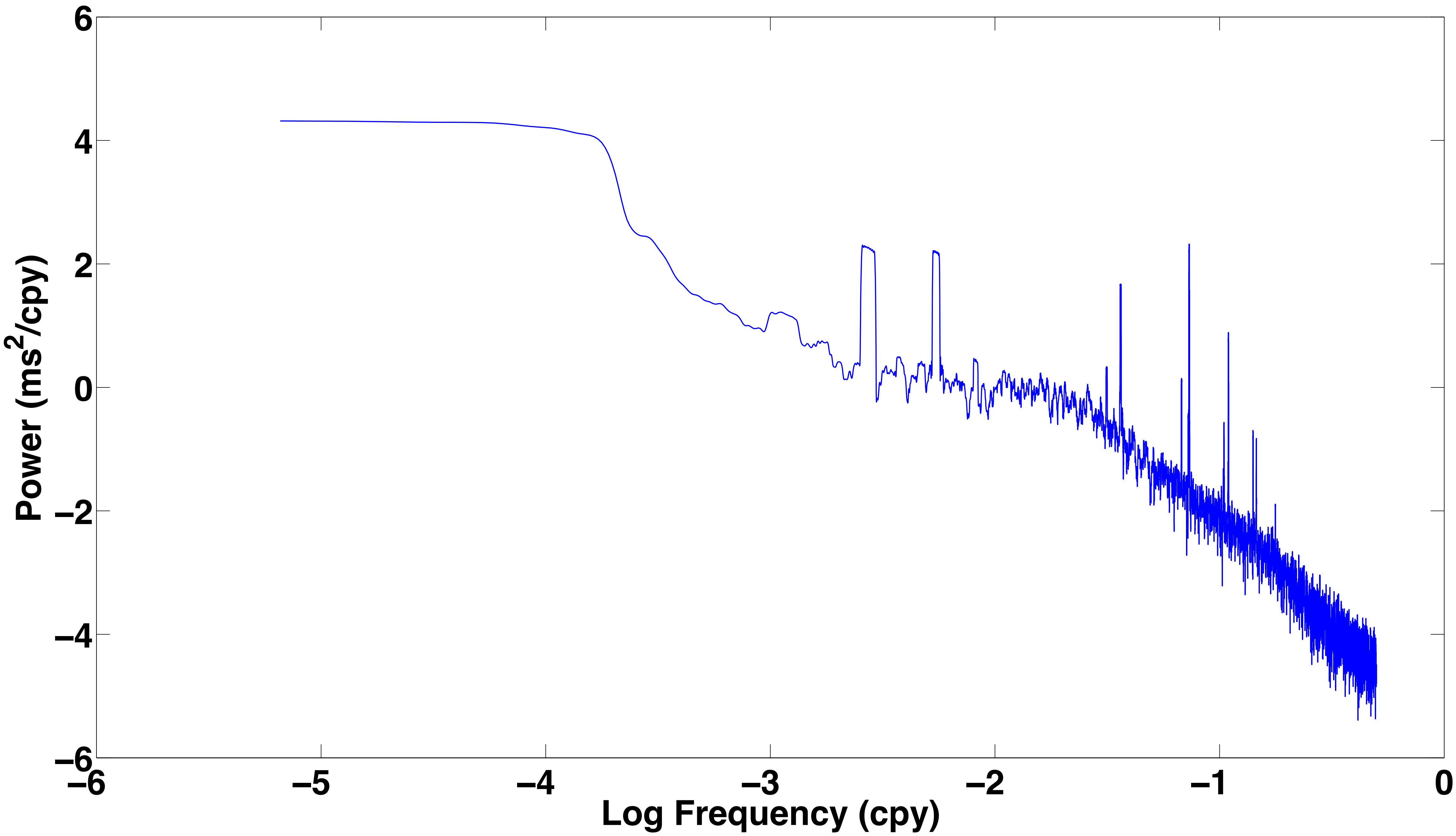
- Daily measurements of the change in length of day from 1962-01-01 thru 2013-12-31



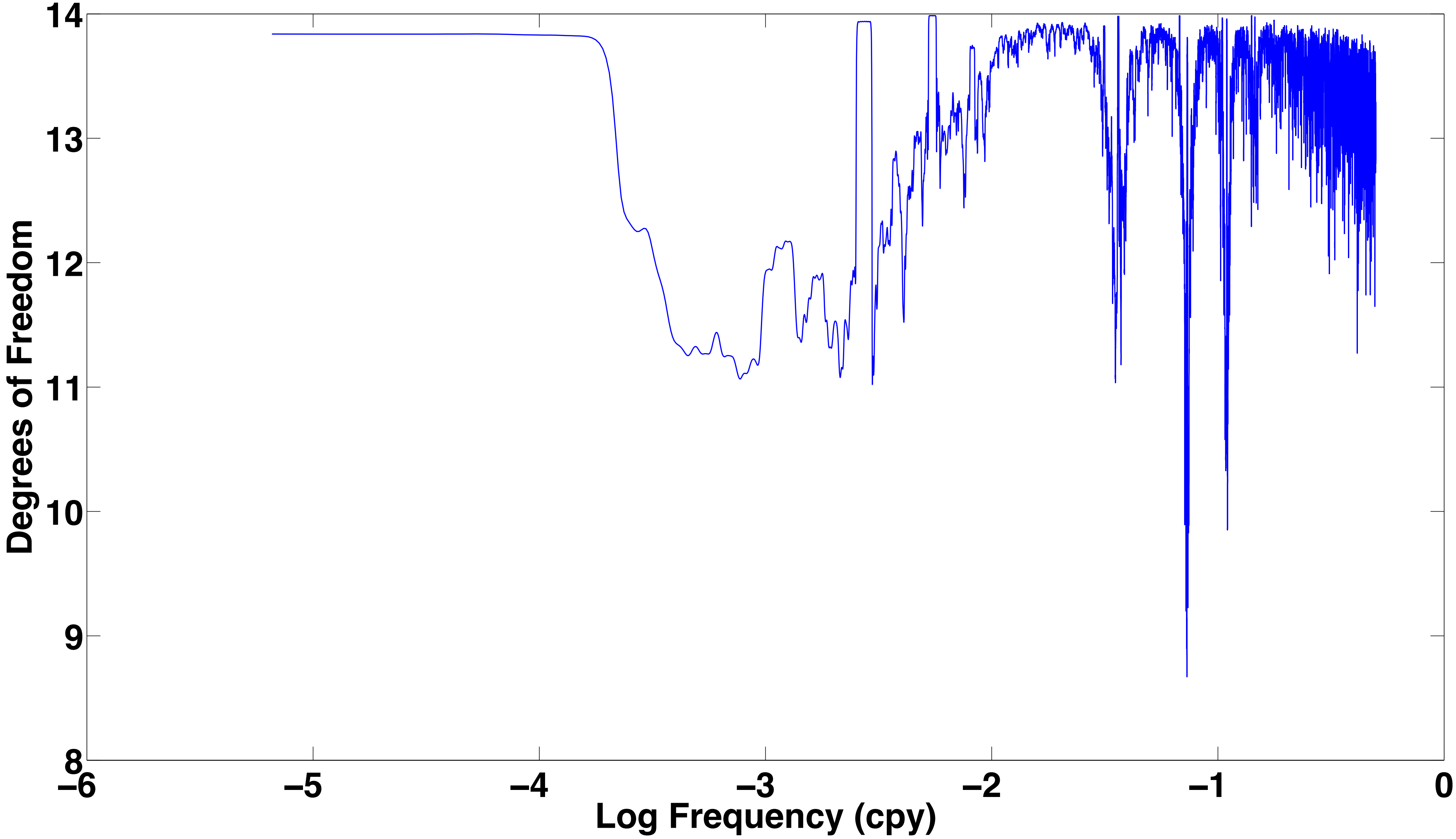


0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
δ lod (ms)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time (d)
x 10^4

Length of Day

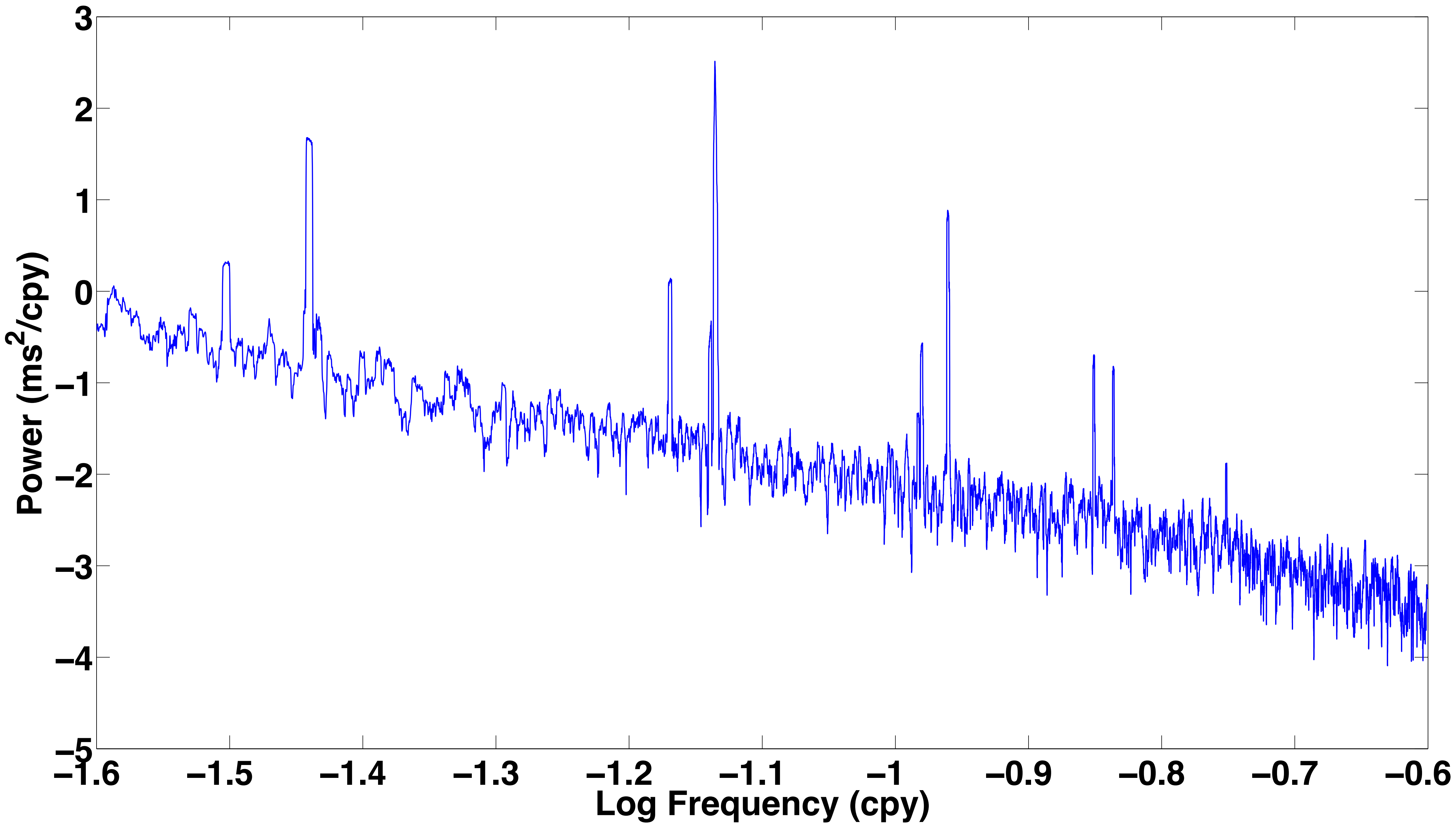


TBW=4 K=7

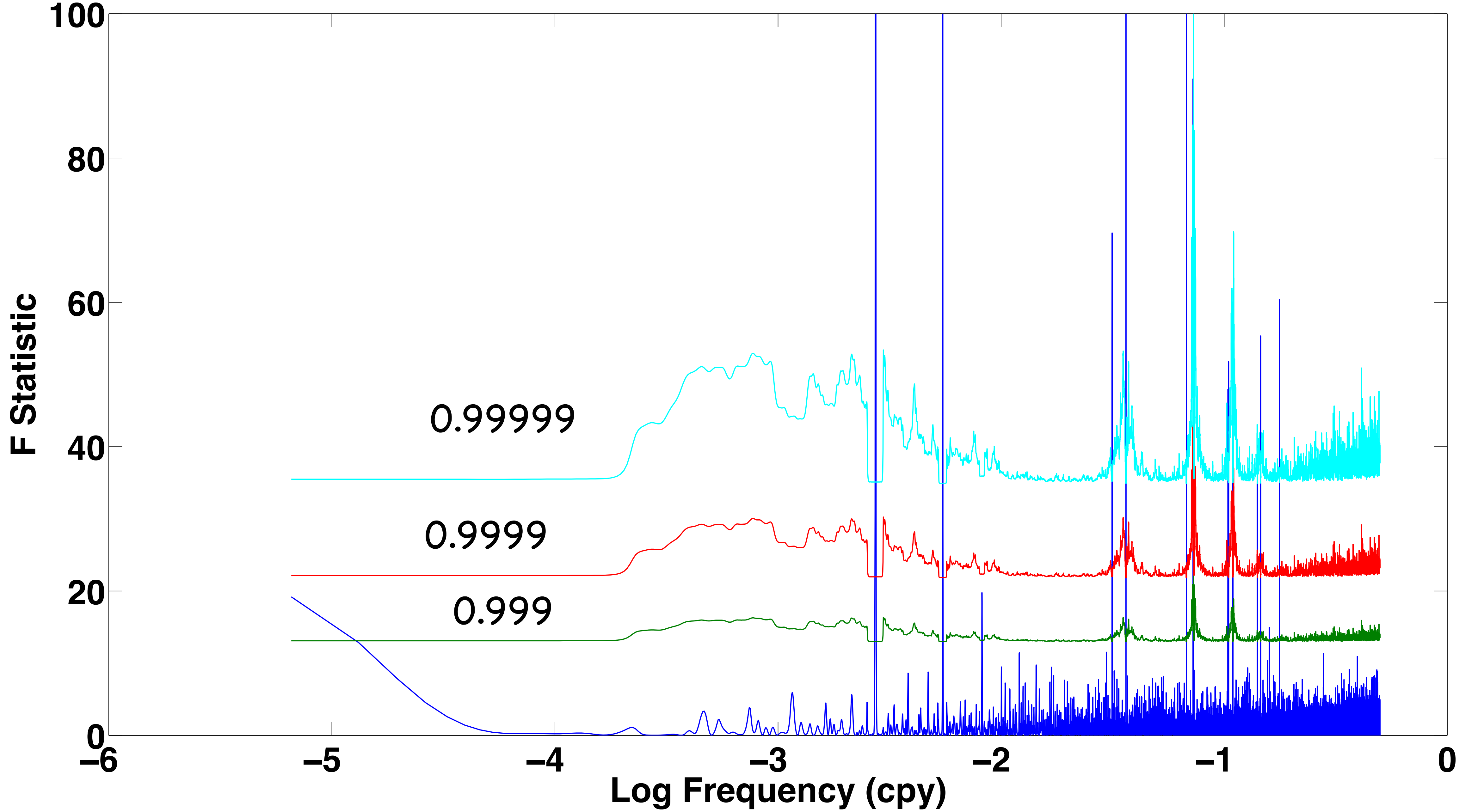


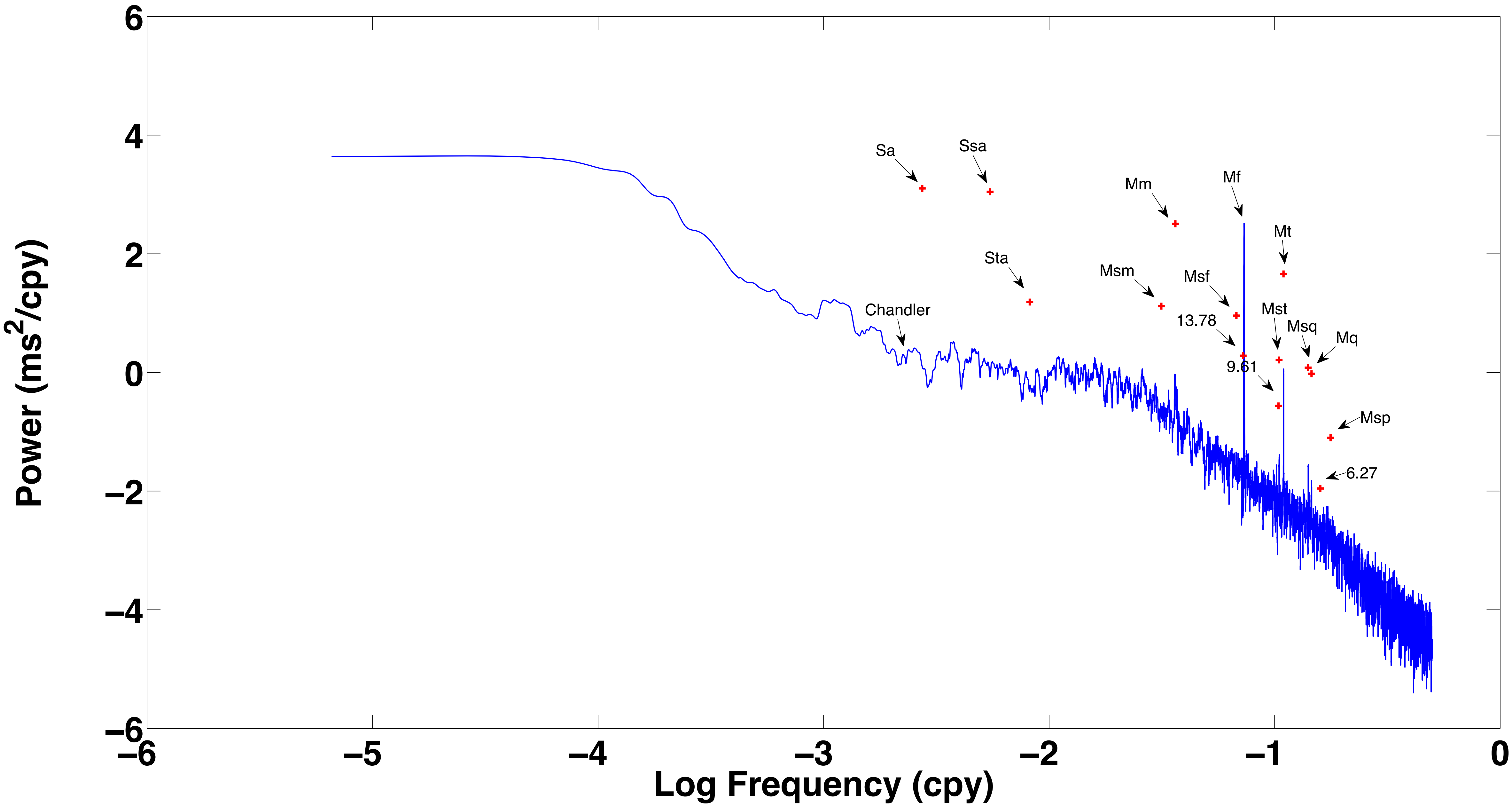
Degrees of Freedom
Log Frequency (cpy)
14
13
12
11
10
9
8
−6
−5
−4
−3
−2
−1
0

T < 1 month



Power (ms^2/cpy)
3
2
1
0
−1
−2
−3
−4
−5
−1.6
−1.5
−1.4
−1.3
−1.2
−1.1
−1
−0.9
−0.8
−0.7
−0.6
Log Frequency (cpy)


100
80
60
40
20
0
F Statistic
0.99999
0.9999
0.999
−6
−5
−4
−3
−2
−1
0
Log Frequency (cpy)



Power (ms^2/cpy)
Log Frequency (cpy)
6
4
2
0
−2
−4
−6
−5
−3
−1
Sa
Ssa
Sta
Chandler
Mm
Msm
Mf
Msf
13.78
9.61
Mt
Mst
Msq
Mq
Msp
6.27

Extensions

- Bivariate and multivariate
- Irregular sampling
- Nonstationary processes

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A multitaper spectral estimator for time-series with missing data

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SUMMARY

A multitaper estimator is proposed that accommodates time-series containing gaps without using any form of interpolation. In contrast with prior missing-data multitaper estimators that force standard Slepian sequences to be zero at gaps, the proposed missing-data Slepian sequences are defined only where data are present. The missing-data Slepian sequences are frequency independent, as are the eigenvalues that define the energy concentration within the resolution bandwidth, when the process bandwidth is $[-1/2, \ 1/2)$ for unit sampling and the sampling scheme comprises integer multiples of unity. As a consequence, one need only compute the ensuing missing-data Slepian sequences for a given sampling scheme once, and then the spectrum at an arbitrary set of frequencies can be computed using them. It is also shown that the resulting missing-data multitaper estimator can incorporate all of the optimality features (i.e. adaptive-weighting, F-test and reshaping) of the standard multitaper estimator, and can be applied to bivariate or multivariate situations in similar ways. Performance of the missing-data multitaper estimator is illustrated using length of day, seafloor pressure and Nile River low stand time-series.

Key words: Fourier analysis; Numerical approximations and analysis; Statistical methods; Time-series analysis.

Matlab code available on Mathworks website

Nonstationary Process

Stationary Process

$$E[dZ(f_1)dZ(f_2)] = S(f_1)\delta(f_1 - f_2)df_1df_2$$

Nonstationary Process

$$E[dZ(f_1)dZ(f_2)] = S_L(f_1, f_2)df_1df_2$$

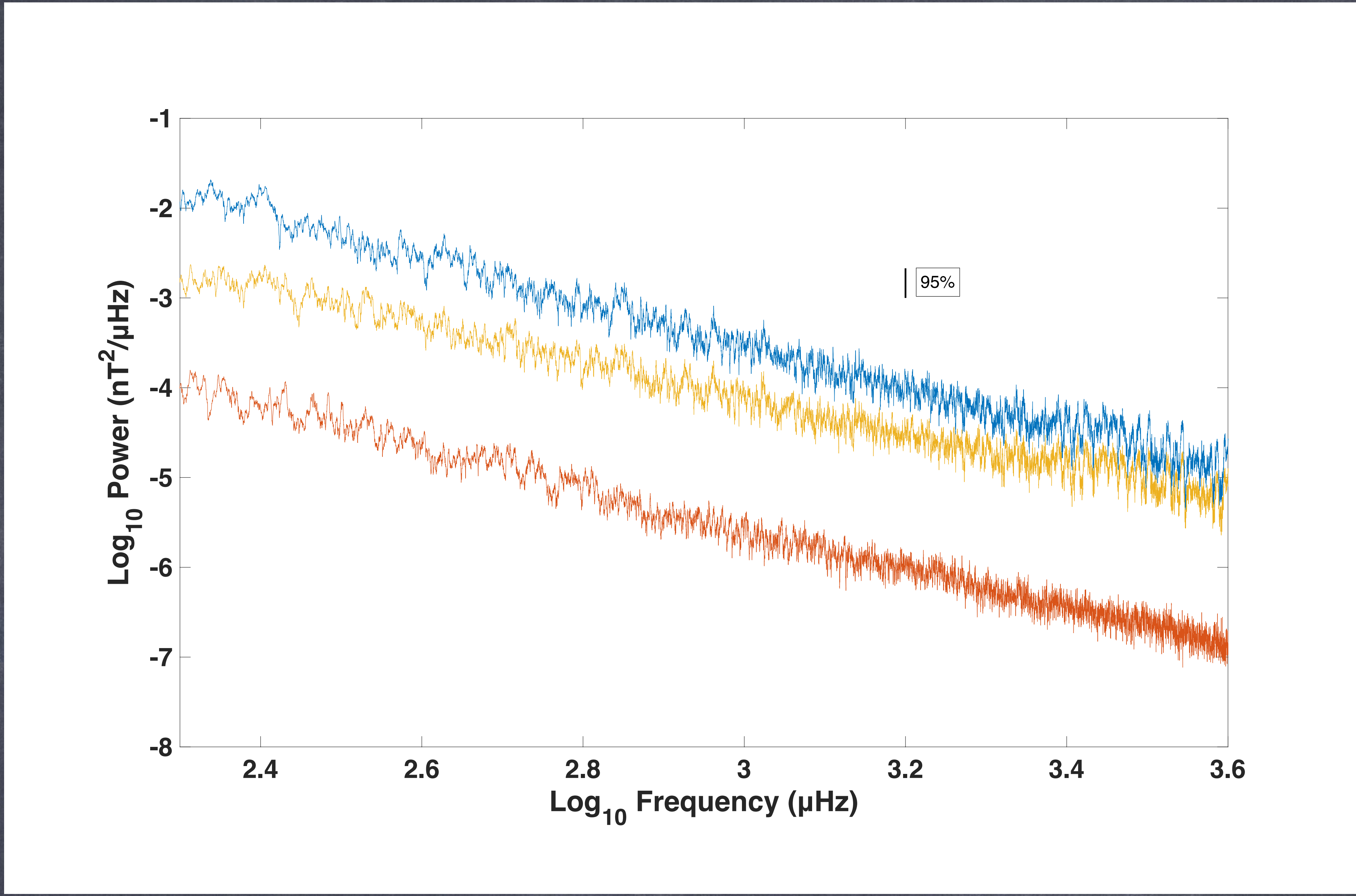
S_L is the Loève spectrum

A non stationary system forced at a given frequency will redistribute power to other frequencies, and the correlation of the spectrum at the two frequencies will be high

Geomagnetic Data

- Honolulu Observatory 2001-2
- Compute standardized spectrum obtained by post-whitening by fitting and removing a quadratic polynomial from the MT result
- Compute coherence versus both ordinary and offset frequency and plotted conditional on its true value being zero, meaning no nonstationarity

H
Z
D



Multitaper spectrum avg for three 60 d sections of data with TBW=5 K=9

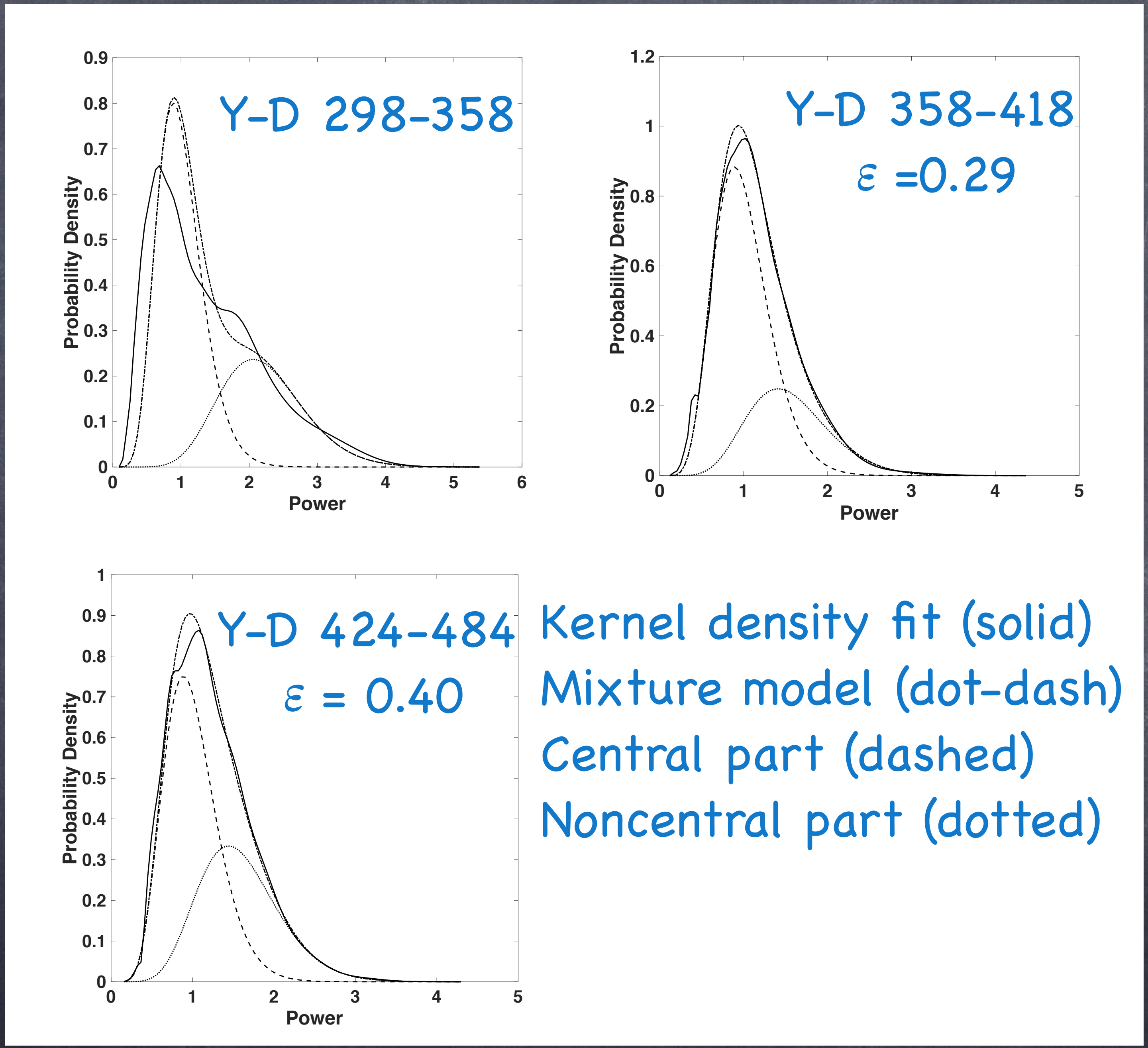
Note enhanced variability over 2000-4000 μHz

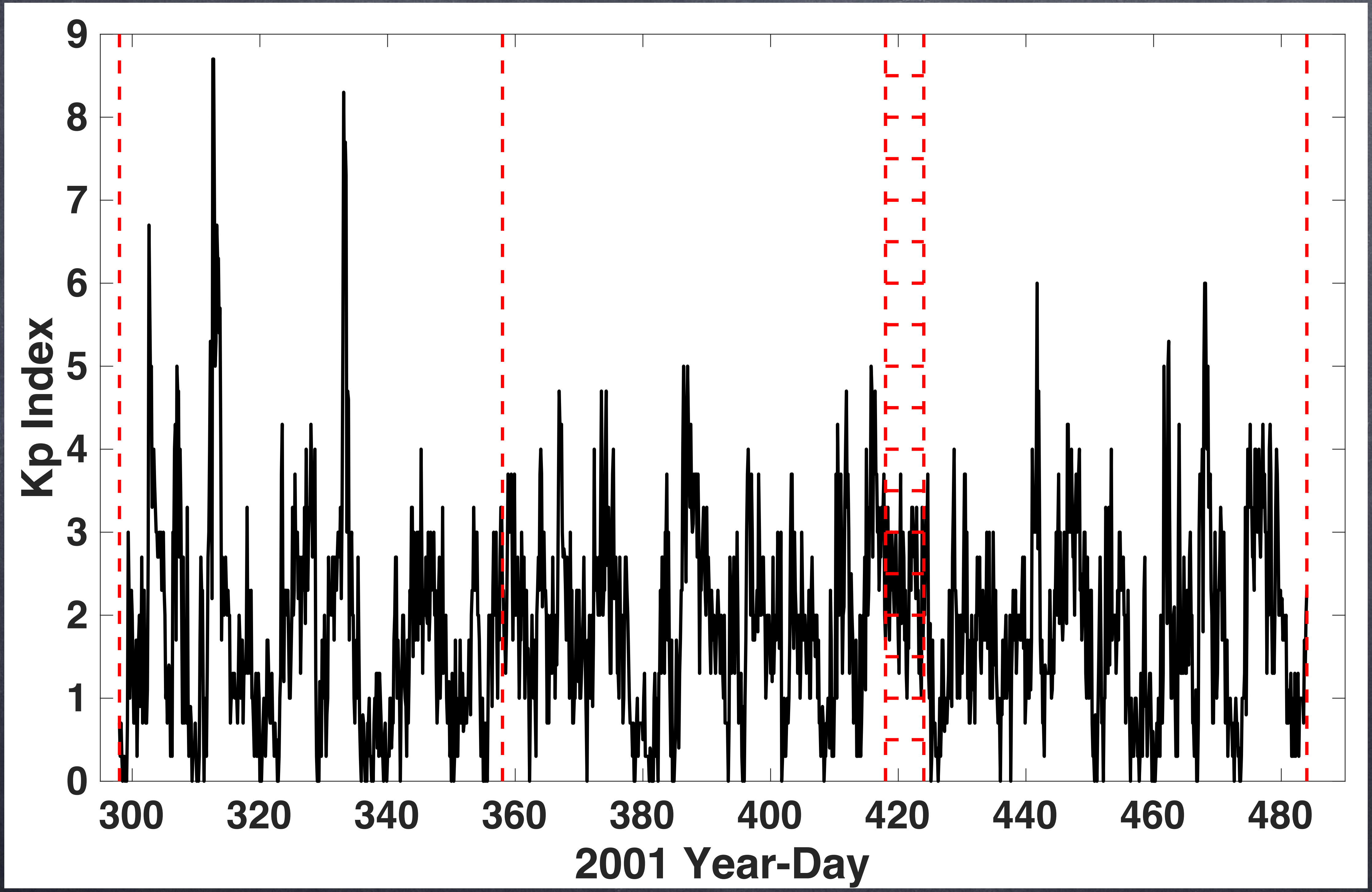
(log of 3.3-3.6, periods of 250-500 s)

Solar Normal Modes

- Represented by quantum numbers n, l, m
- Characterized by central frequency and Q
- Pressure modes $p_{n,l,m}$ over 250-5100 μHz
- Excited by turbulence -> amplitudes are random
- Qs of several thousand
- Persistence for a couple of months

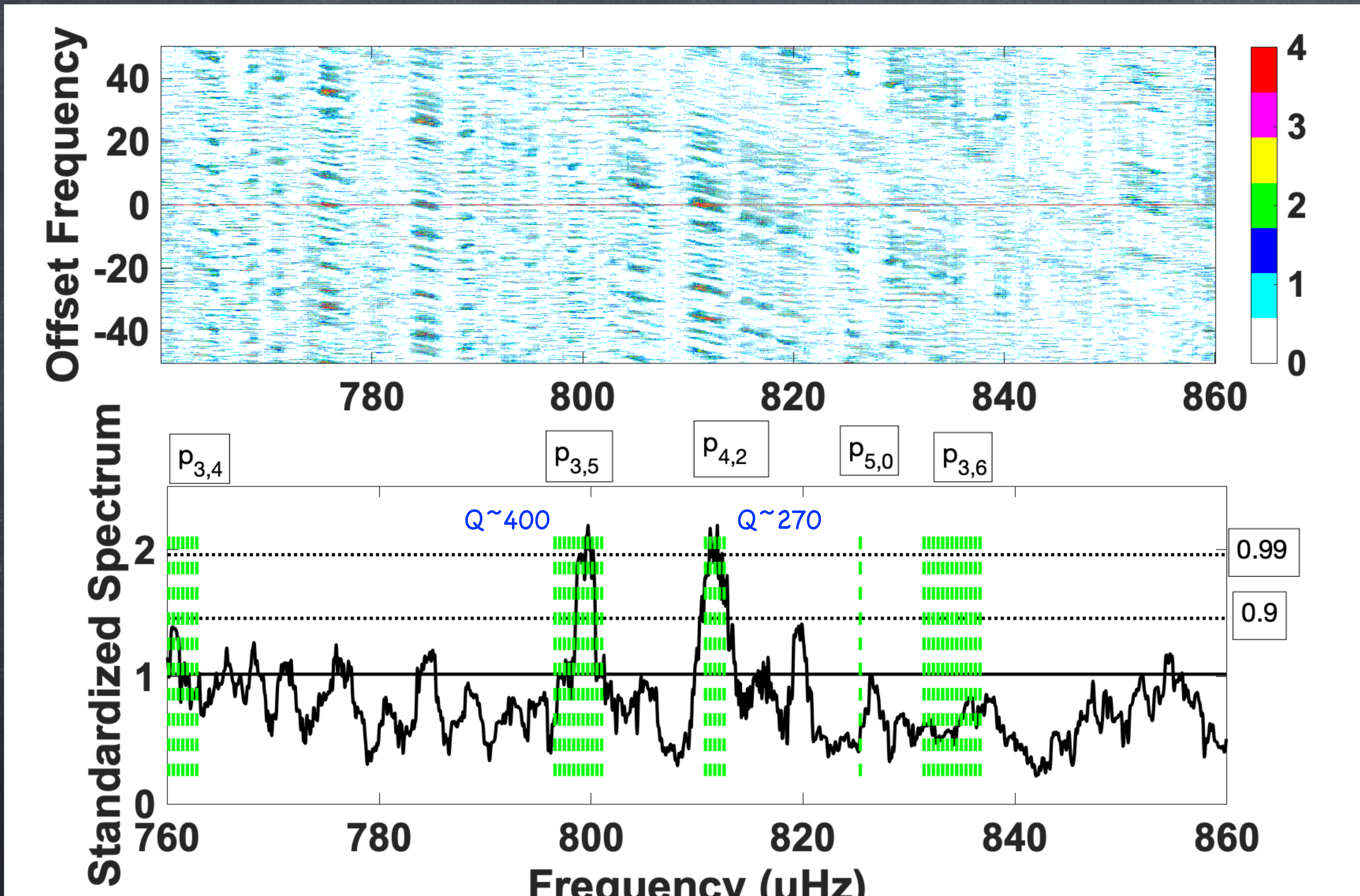
Mixture Noncentral/Central χ^2 Fit over 2000-3000 μHz

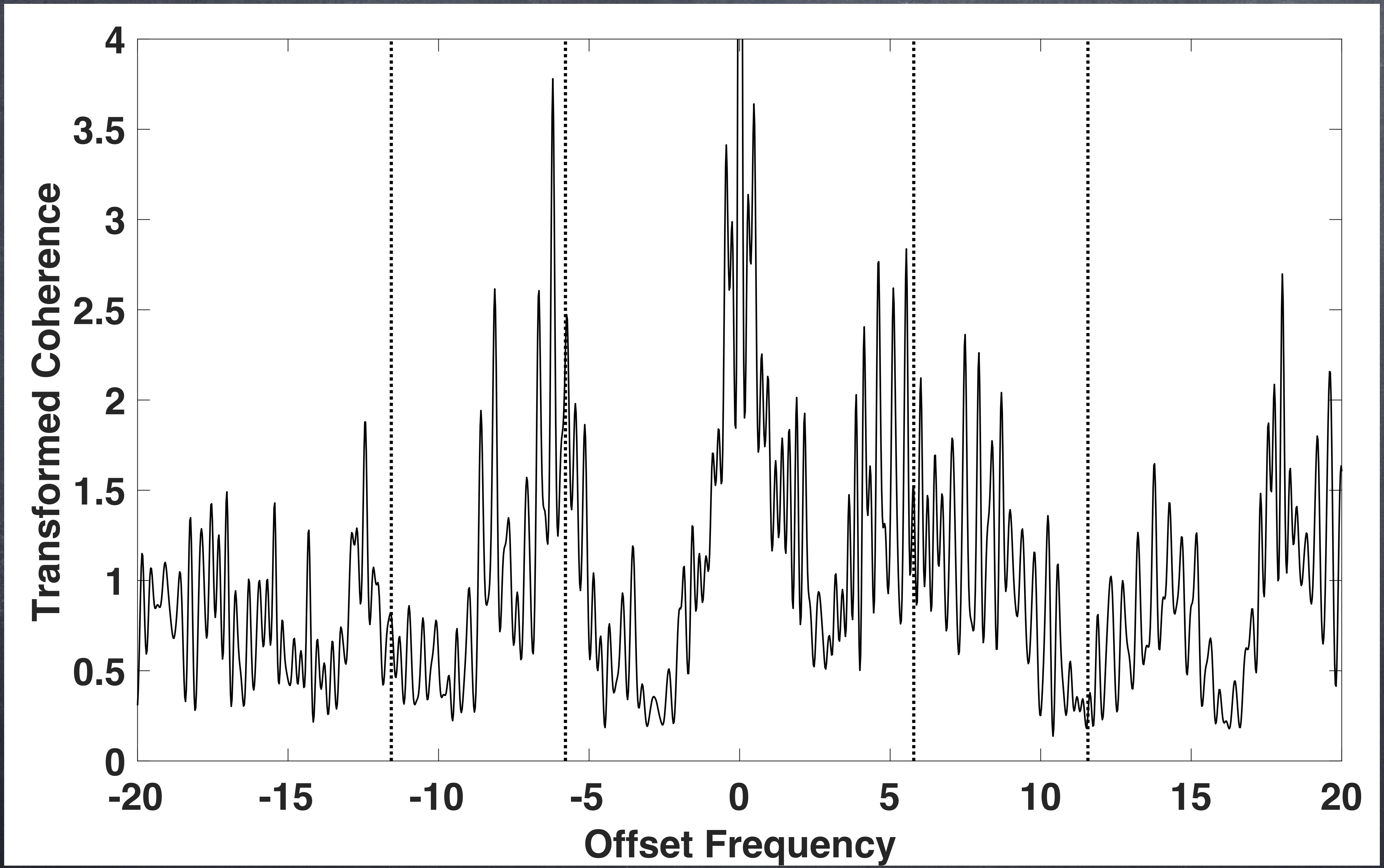




Kp Index
9
8
7
6
5
4
3
2
1
0
300
320
340
360
380
400
420
440
460
480
2001 Year-Day

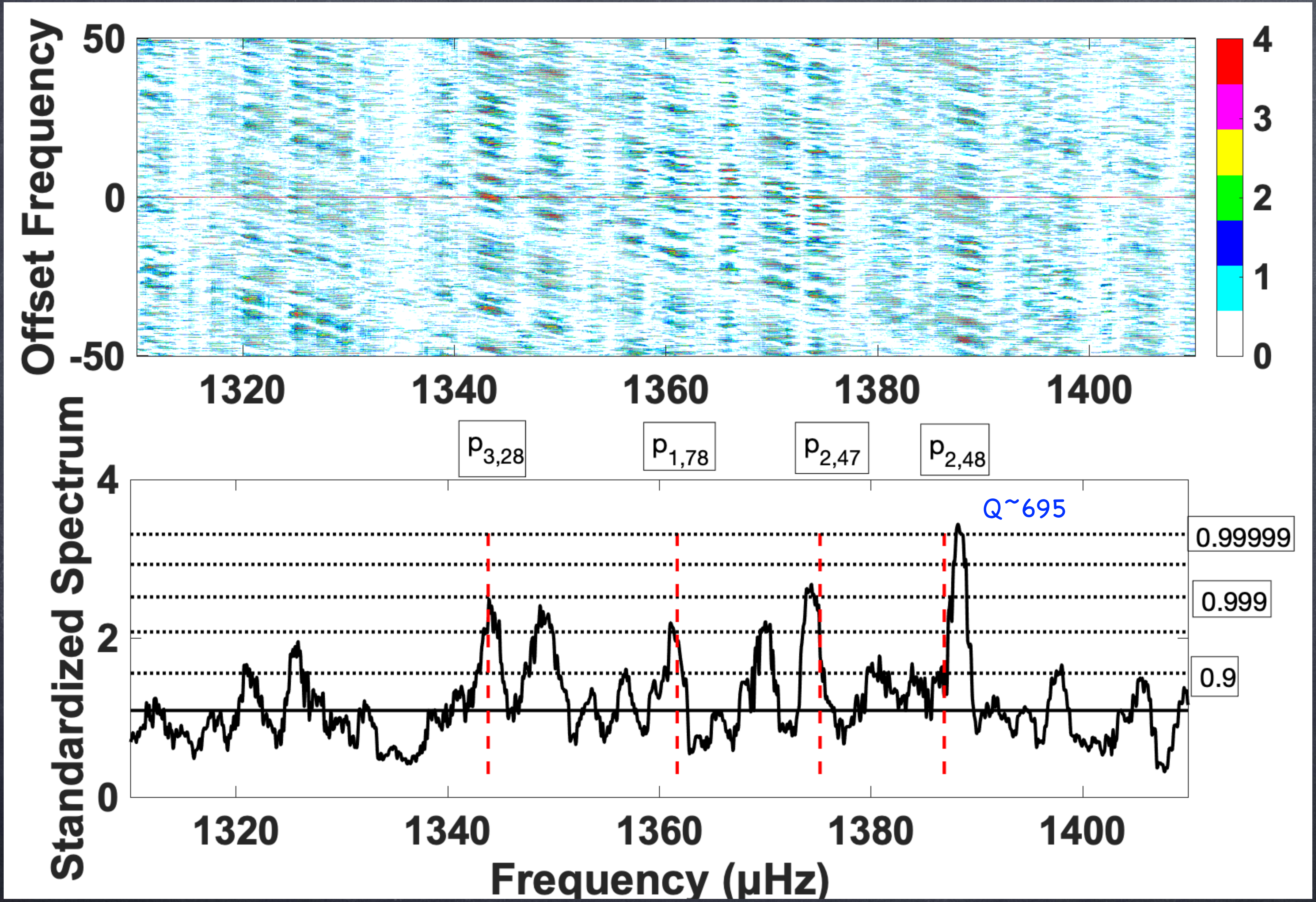
H component for Y-D 424-484



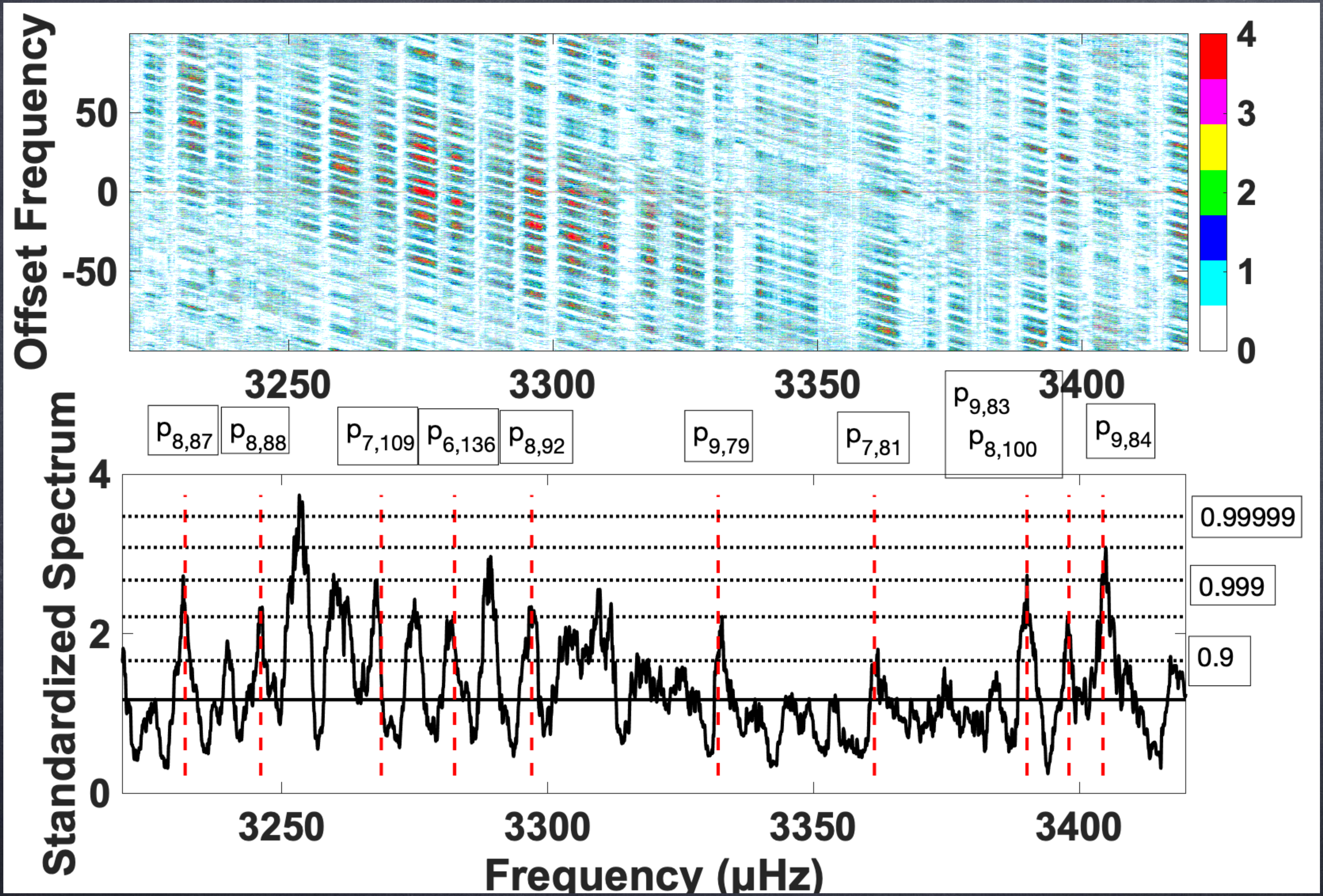


Transformed Coherence
Offset Frequency
4
3.5
3
2.5
2
1.5
1
0.5
0
-20
-15
-10
-5
0
5
10
15
20

H component for Y-D 424-484



H component for Y-D 358-418



Lecture notes on MT achave@whoi.edu

Maxwell will take questions